OPTIMIZATION METHODS: CLASS 4

Polytopes and their properties

The exercises are on the next page.

D: A set $K \subseteq \mathbb{R}^d$ is a *convex set*, if $\forall x, y \in K, \forall t \in [0,1] : tx + (1-t)y \in K$. In other words, if you take two points inside the convex set K, the entire line segment between those two points must belong to K.

D: A *hyperplane* is any affine space in $\mathbb{R}^d$ of dimension $d-1$. Thus, on a 2D plane, any line is a hyperplane. In the 3D space, any plane is a hyperplane, and so on.

A hyperplane splits the space $\mathbb{R}^d$ into two *halfspaces*. We count the hyperplane itself as a part of both halfspaces.

D: A *convex polytope* is any object in $\mathbb{R}^d$ that is an intersection of finitely many halfspaces. Alternatively, we can say that a convex polytope is any set of points of the form $\{x | Ax \leq b\}$ for some real matrix A and some real vector b.

We are not going to be investigating non-convex polytopes in this class, so we will often say just *polytope* for short.

D: We say that a convex polytope P is *bounded* if P can be contained inside a ball of a fixed radius. In other words, it does not stretch infinitely in some direction and it has finite volume.

Note: Sometimes in the literature, people distinguish between *convex polyhedra* which can be unbounded and *convex polytopes* which are always bounded. Since many people (including myself) often get those names mixed up, we will prefer the terms "polytope" and "bounded polytope".

D: Let P be some convex polytope in $\mathbb{R}^d$. We say that a hyperplane H is a *supporting hyperplane* if it does not cut the polytope.

In other words, if the hyperplane H is defined as $\{x \in \mathbb{R}^d | c^T x = t\}$, then we say H is supporting if and only if it holds that $\forall y \in P : \{c^T y \leq t\}$ or it holds that $\forall y \in P : \{c^T y \geq t\}$.

D: A *face* F of a polytope P is any set of the form $F = P \cap H$ for any supporting hyperplane H.

Note that our definition allows that $P \cap H = \emptyset$. We also count P itself as a face. These two faces $P, \emptyset$ are *improper* faces, the rest of the faces (whenever $P \cap H \neq \emptyset$) are called *proper*.

D: A *vertex* of a polytope P is a face of dimension 0 (a single point). An *edge* is any face of dimension 1 (a line segment, half-line or a line). On the other side of the spectrum, a *facet* of P is a face of dimension $d-1$.

D: A d-dimensional *simplex* is a polytope which arises as a convex hull of any $d+1$ affinely independent points. All simplexes are structurally the same, so whenever you think about a simplex, you can consider the set:

$$\operatorname{conv}(0, (0,0,\ldots,0,1), (0,0,\ldots,1,0), \ldots, (1,0,\ldots,0,0))$$

T(Vertex description of a polytope**):** Every bounded convex polytope is equal to the convex hull of all its vertices. Bounded polytopes therefore can be described using all their halfspaces (then the polytope is their intersection) or their vertices (then the polytope is their convex hull).

T(Basic solutions are exactly vertices of a polytope**):** A point x_i is a vertex of a convex polytope $Ax \leq b$ defined in $\mathbb{R}^d$ if and only if x_i is a *basic solution* of $Ax \leq b$.

In other words: In order for x_i to be a vertex, it has to be inside the polytope (satisfy all the inequalities) and it also must satisfy d *linearly-independent* inequalities with equality.

Exercise one **Two convex properties:**

- Prove the following: if each point $\vec{x_1}, \vec{x_2}, \ldots, \vec{x_n} \in \mathbb{R}^d$ satisfies a set of constraints $\vec{a_i}^T \cdot \vec{x_j} \leq b_i$ for $i, j \in \{1, 2, \ldots, n\}$, then any convex combination of the points x_i satisfies the same set of constraints. In other words, $\forall \alpha_1, \ldots, \alpha_n \geq 0$ such that $\sum_{j=1}^{n} \alpha_j = 1$ it holds that

$$\vec{a_i}^T \cdot \left(\sum_{j=1}^{n} \alpha_j \vec{x_j} \right) \leq b_i.$$

- Prove the following: if a set of points $\vec{x_1}, \vec{x_2}, \ldots, \vec{x_n} \in \mathbb{R}^d$ satisfies a set of constraints $\vec{a_i}^T \cdot \vec{x_j} \leq b_i$ for $i, j \in \{1,\ 2, \ldots, n\}$, then the same set of points x_i satisfies any *convex combination of the constraints*. Formally, prove that $\forall \beta_1, \ldots, \beta_n \geq 0$ such that $\sum_{i=1}^{n} \beta_i = 1$ it holds that:

$$\left(\sum_{k=1}^{n} \beta_k \vec{a_k} \right)^T \cdot \vec{x_j} \leq \sum_{k=1}^{n} \beta_k b_k.$$

Hint: Notice the difference in the statements! Also, prove both statements for the smallest set of things for which it makes sense: prove the first for many points and one inequality, and prove the second for one point and many inequalities. Then argue that this is enough.

Exercise two Prove that a set of all optimal solutions of an LP, for instance one of this form: $\max c^T x, Ax \leq b, x \geq 0$ is a convex set.

Exercise three **Properties of convex polytopes:**

- Prove that a face F of a polytope P is a polytope itself; also prove that every face G of F is also a face of P.
- Let now us suppose that P is *bounded* (it has no infinite face). Using this, prove that every face F of a polytope P is a convex hull of some subset of vertices of P.
- Prove that any face of a simplex is a simplex itself.
- Think of your favourite polytope P and find two different supporting hyperplanes h_a, h_b which intersect P and induce the same face F.

Exercise four Check if the point $v = (1, 1, 1, 1)$ is a vertex of a polytope P defined as the following set of inequalities:

$$\begin{pmatrix} -1 & -6 & 1 & 3 \\ -1 & -2 & 7 & 1 \\ 0 & 3 & -10 & -1 \\ -6 & -11 & -2 & -12 \\ 1 & 6 & -1 & -3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \leq \begin{pmatrix} -3 \\ 5 \\ -8 \\ -7 \\ 4 \end{pmatrix}$$

Exercise five Prove that any bounded convex polytope of dimension d in $\mathbb{R}^d$ has at least $d + 1$ vertices and at least $d + 1$ facets.

Exercise six Find all vertices of a polytope defined as follows:

$$\begin{aligned} 2x_1 + x_2 + x_3 &\leq 14 \\ 2x_1 + 5x_2 + 5x_3 &\leq 30 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \\ x_3 &\geq 0 \end{aligned}$$