OPTIMIZATION METHODS: CLASS 3

Linearity, convexity, affinity

The exercises are on the opposite side.

D: A set $A \subseteq \mathbb{R}^d$ is an *affine space*, if A is of the form $L + v$ for some linear space L and a shift vector $v \in \mathbb{R}^d$. By "A is of the form $L+v$" we mean a bijection between vectors of L and vectors of A given as $b(u) = u + v$. Each affine space has a *dimension*, defined as the dimension of its associated linear space L.

D: A vector x is an *affine combination* of a finite set of vectors $a_1, a_2, \dots a_n$ if $x = \sum_{i=1}^{n} \alpha_i a_i$, where α_i are real number satisfying $\sum_{i=1}^{n} \alpha_i = 1$.

A set of vectors $V \subseteq \mathbb{R}^d$ is *affinely independent* if it holds that no vector $v \in V$ is an affine combination of the rest.

D: GIven a set of vectors $V \subseteq \mathbb{R}^d$, we can think of its *affine span*, which is a set of vectors A that are all possible affine combinations of any finite subset of V.

Similar to the linear spaces, affine spaces have a finite basis, so we do not need to consider all finite subsets of V, but we can generate the affine span as affine combinations of the base.

D: A *hyperplane* is any affine space in $\mathbb{R}^d$ of dimension $d-1$. Thus, on a 2D plane, any line is a hyperplane. In the 3D space, any plane is a hyperplane, and so on.

A hyperplane splits the space $\mathbb{R}^d$ into two *halfspaces*. We count the hyperplane itself as a part of both halfspaces.

D: A set $K \subseteq \mathbb{R}^d$ is a *convex set*, if $\forall x, y \in K, \forall t \in [0,1] : tx + (1-t)y \in K$. In other words, if you take two points inside the convex set K, the entire line segment between those two points must belong to K.

D: A vector x is a *convex combination* of a set of vectors $a_1, a_2, \dots a_n$ if $x = \sum_{i=1}^{n} \alpha_i a_i$, where α_i are real numbers satisfying $\sum_{i=1}^{n} \alpha_i = 1$ and also $\forall i : \alpha_i \in [0,1]$.

A set of vectors/points $V \subseteq \mathbb{R}^d$ is *in a convex position*, if it holds that no vector $v \in V$ is a convex combination of the rest.

D: As with linearity and affinity, for convexity we also define a span/hull:

If we have a set of vectors $V \subseteq \mathbb{R}^d$, its *convex hull* is a set of all vectors C, which are convex combinations of any finite subset of the vectors in V.

Here, we really need to consider any finite subset of V, because convex sets in general do not have a finite basis.

D: A *convex polytope* is any object in $\mathbb{R}^d$ that is an intersection of finitely many halfspaces. Alternatively, we can say that a convex polytope is any set of points of the form $\{x | Ax \leq b\}$ for some real matrix A and some real vector b.

A quick reminder from linear algebra:

T: Every linear space of dimension k contains a basis of k vectors. We can find a special basis that is *orthogonal* or even *orthonormal*). And for any basis (even a non-orthogonal one) we can compute its *orthogonal complement.* (How?)

From last time:

Exercise one Josef K. got an exercise at his Optimization methods class:

Design an integer program for the travelling salesman problem: For a given graph with distances $G = (V, E, f)$, where $f : E \to \mathbb{R}_0^+$, find a Hamiltonian cycle with the shortest length.

He suggests the following:

"For every edge uv we have a variable $x_{uv} \in \{0, 1\}$, the target function is $\min \sum_{uv \in E} f(uv)x_{uv}$ and for every vertex u we create a condition of the form $\sum_{i|ui \in E} x_{ui} = 2$."

Prove that Josef K. got the right solution – or prove him wrong and suggest a better one.

Exercise two Let us consider a polytope (actually, a line segment)

$$P = \{x \in \mathbb{R} | x \geq 1 \& x \leq 2\}.$$

Transform its inequalities into an equational form and then draw the polytope in the equational form.

Exercise three Prove the following equivalence:

A set of $n+1$ vectors $v_0, v_1, v_2, v_3, \ldots, v_n$ in $\mathbb{R}^d$ is affinely independent if and only if the set of n vectors $v_1 - v_0, v_2 - v_0, v_3 - v_0, \ldots, v_n - v_0$ is linearly independent.

Exercise four Alice and Bob play a game. Alice will think of a linear inequality in $\mathbb{R}^3$ but it will not describe it to Bob. She will only tell Bob three points b_1, b_2, b_3 in $\mathbb{R}^3$, which satisfy the inequality.

Bob now must call out new points $b_4, b_5, b_6 \ldots$ which also satisfy the inequality – until Alice gets bored of the game and they both go play hopscotch.

Suggest a strategy for Bob to win.

Exercise five

1. Can two 2D planes intersect in exactly one point, if we place them in $\mathbb{R}^4$?
2. *If that was too easy:* Can two $3D$ spaces (affine spaces of dimension 3) intersect in exactly one point in $\mathbb{R}^5$?

Exercise six

1. Prove that each affine space can be expressed as an intersection of finitely many affine hyperplanes.
2. Prove that every hyperplane can be expressed as the set $\{x | c^T x = b\}$.

Hint: Whenever you want to prove something about an affine space, try shifting it using the vector $-v$ so that it becomes a linear space L, and then argue about the linear space instead.

Exercise seven Prove the following equivalence, which gives you an easy way to algebraically describe affine spaces:

A set $F \subseteq \mathbb{R}^d$ is an affine space if and only if $F = \{x \in \mathbb{R}^d | Ax = b\}$ for some matrix $A \in \mathbb{R}^{d \times d}$ and some vector $b \in \mathbb{R}^d$. (Just to make things simpler, we count an empty set as an affine space too.)

Exercise eight We know that a set K is convex if the set contains all line segments with endpoints in K. Prove a very similar description for affinity:

A set A is an affine subspace of R^d if and only if for each two points $a, b \in A$ the entire *line* defined by a, b is contained in A.