OPTIMIZATION METHODS: CLASS 1

what we need from other courses

The class "Optimization Methods" is mostly about studying linear programming and its applications. Before we start with it, we first dust off knowledge from some other courses, like:

Linear algebra

EXERCISE ONE Use Gauss-Jordan elimination to solve the following system of equations:

$$\begin{aligned} x + 2y + 3z + w &= 0 \\ 2x + 4y + z + 2w &= 0 \\ x + 2y - 2z + w &= 0 \\ 5z &= 0 \end{aligned}$$

EXERCISE TWO Graphically solve the following system:

$$\begin{aligned} x + y + z &\geq 2 \\ x + y + z &\leq 2 \\ x + 2y - z &\leq 10 \\ x &\geq 0 \\ y &\geq 0 \\ z &\geq 0 \end{aligned}$$

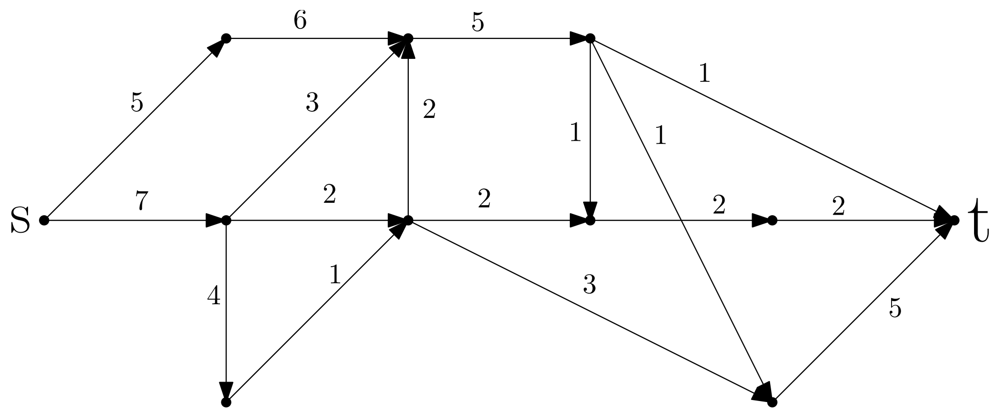
Point out the solutions that maximize x, y, z, respectively.

Hint: By "graphically solve" we simply mean "draw the inequalities in 3D, look at the intersecting area and decide whether it is empty (no solution), one-point, limited and so on".

Algorithms and data structures

EXERCISE THREE

Find the minimum s, t-cut for the following directed graph:



EXERCISE FOUR Let us consider the problem VERTEX COVER, defined as follows:

Input: Undirected graph G.

Goal: To find a set of vertices X so that each edge $e \in E(G)$ has at least one endpoint in X. From all such sets, we look for the smallest in its size.

Find a 2-approximation algorithm for VERTEX COVER.

Combinatorics and graph theory

EXERCISE FIVE Consider an undirected bipartite graph G_1 and an undirected arbitrary graph G_2. Which algorithm does one use for the maximum matching in G_1? And do you know an algorithm that one can use for G_2? If you can, recall the worst-case time complexity as well.

EXERCISE SIX You probably remember that the dual problem for a s,t-maximum-flow is a s,t-minimum cut. Do other problems have duals too? Consider the problem of the shortest s,t-path in an undirected, unweighted graph G, and suggest a dual problem for this.

Hint: This is a creative exercise. We will find such a precise dual later in the class, but this is not the goal here – just use your imagination.

Linear programming formulations

EXERCISE SEVEN A bakery produces four things: bread, bagels, baguettes and donuts. To bake a single bread, they need 500g of flour, 10 eggs and 50 grams of salt. To bake a bagel, they need 150 grams of flour, 2 eggs and 10g of salt. For a baguette, they need 230g of flour, 7 eggs and 15g of salt. For a donut, they need 100g of flour and 1 egg.

The bakery has a daily supply of 5 kg of flour, 125 eggs, and 500g of salt.

The bakery charges 20 CZK for one bread, 2 CZK for a bagel, 10 CZK for a baguette and 7 CZK for a donut. The bakery tries to maximize its profit. Formulate a linear program that suggests the right amount of bread, bagels, baguettes and donuts that the bakery should produce. (You don't have to solve it!)

EXERCISE EIGHT

Part 1. Suppose we have a system of linear inequalities that also contains sharp inequalities. One that may look like this:

$$\begin{aligned} 5x + 3y &\leq 8 \\ 2x - 5z &< -3 \\ 6x + 5y + 2w &= 5 \\ 3z + 2w &> 5 \\ x, y, z, w &\geq 0 \end{aligned}$$

Is there a way to check if this system has a feasible solution using a linear program?

Part 2. Does this mean that linear programming allows strict inequalities? Not really. As a strange example, construct a "linear program with a strict ineqality" that satisfies the following:

- There is a simple finite upper bound on its optimum value;
- There is a feasible solution;
- There is no optimal solution.

This may not happen for a linear program – for a bounded LP, once there exists a feasible solution, there exists also an optimal solution.

EXERCISE NINE Let us have a non-standard linear program of the form: $\max c^T x$ subject to $Ax \leq b$, but $x \in \mathbb{R}^n$, not just $x \geq 0$. Is it possible to transform this program into an equivalent program of the canonical form, that is $\max d^T x'$ subject to $A'x' \leq b$ and $x' \geq 0$?