Mathematical Analysis I

Exercise sheet 8

3 December 2015

References: Abbott 4.2, 4.3. Bartle & Sherbert 4.1, 4.2, 5.1, 5.2

1. Let $f : A \to \mathbb{R}$ and let c be a limit point of the domain A.
   Define what is meant by $\lim_{x\to c} f(x) = L$.
   (i) Prove that if $f(a_n) \to L$ for all sequences (a_n) of points in $A \setminus \{c\}$ convergent to limit c then $\lim_{x\to c} f(x) = L$.
   (ii) Prove the converse statement to (i).
   (iii) Deduce from (ii) that if there is a sequence (a_n) of points in $A \setminus \{c\}$ convergent to limit c such that the sequence $(f(a_n))$ is divergent then $\lim_{x\to c} f(x)$ does not exist.
   (iv) As a special case of (iii) deduce that if there are two sequences (a_n) and (b_n) of points in $A \setminus \{c\}$ convergent to limit c but such that $\lim f(a_n) \neq \lim f(b_n)$ then $\lim_{x\to c} f(x)$ does not exist.
   (v) Use (iv) to show that the signum function $\operatorname{sgn} : \mathbb{R} \to \mathbb{R}$ defined by
   $$\operatorname{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$
   does not have a limit as $x \to 0$.

2. Show that the following limits do not exist:
   (i) $\lim_{x\to 0} \frac{1}{x^2}$ $(x > 0)$
   (ii) $\lim_{x\to 0} \frac{1}{\sqrt{x}}$ $(x > 0)$
   (iii) $\lim_{x\to 0}(x + \operatorname{sgn}(x))$
   (iv) $\lim_{x\to 0} \sin\left(\frac{1}{x^2}\right)$

3. Let $c \in \mathbb{R}$.
   (i) State the Algebraic Limit Theorem for Functional Limits.
   (ii) Show that $\lim_{x\to c} a = a$ for any constant $a \in \mathbb{R}$ and that $\lim_{x\to c} x = c$.
   (iii) Deduce from (ii) that $\lim_{x\to c} x^n = c^n$ for any $n \in \mathbb{N}$.
   (iv) Deduce from the previous parts that if $p(x)$ is a polynomial with real coefficients then $\lim_{x\to c} p(x) = p(c)$.
   (v) If $p(x)$ and $q(x)$ are polynomials with real coefficients, show using the previous results that
   $$\lim_{x\to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}$$
   unless $q(c) = 0$, in which case the limit does not exist.

4. Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ and $f(A) = \{f(x) : x \in A\} \subseteq B$.
Define what it means for f to be *continuous* at a point $c \in A$.

(i) Suppose that f is continuous at c and g is continuous at $f(c) \in B$. Prove that the composition $g \circ f$ is continuous at c.

Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = \sqrt[3]{x}$.

(ii) Show that g is continuous at $c = 0$

(iii) Prove that g is continuous at a point $c \neq 0$.

(iv) Assuming the result of question 3(iv), deduce that $\sqrt[3]{p(x)}$ is continuous on $\mathbb{R}$ for any polynomial $p(x)$ with real coefficients.

5. For each of the following choices of A, construct a function $f : \mathbb{R} \to \mathbb{R}$ which has discontinuities at every point of A and is continuous on the complement $\mathbb{R} \setminus A$:

(i) $A = \mathbb{Z}$

(ii) $A = \{x : 0 < x < 1\}$

(iii) $A = \{x : 0 \leq x < 1\}$

(iv) $A = \{\frac{1}{n} : n \in \mathbb{N}\}$

6. Let $f : \mathbb{R} \to \mathbb{R}$ and assume there is a constant C such that $0 < C < 1$ and

$$|f(x) - f(y)| \leq C|x - y|$$

for all $x, y \in \mathbb{R}$.

(i) Show that f is continuous on $\mathbb{R}$.

(ii) Beginning with an initial value $y_1 \in \mathbb{R}$, define the sequence $(y_n) = (y_1, f(y_1), f(f(y_1)), \dots)$ recursively by setting $y_{n+1} = f(y_n)$. Show that (y_n) is a Cauchy sequence.

(iii) Let $y = \lim y_n$. Prove that y is a fixed point of f (i.e., $f(y) = y$) and that it is the unique fixed point of f (i.e., if $f(y') = y'$ then $y' = y$).

(iv) For an arbitrary initial value $x \in \mathbb{R}$, show that the sequence $(x_n) = (x, f(x), f(f(x)), \dots)$ defined recursively by $x_{n+1} = f(x_n)$ is convergent to the value y defined in (iii).