Mathematical Analysis I

Exercise sheet 5

5 November 2015

References: Abbott 2.5, 2.6. Bartle & Sherbert 3.3, 3.4, 3.5

1. Give an example of each of the following, or argue that no such example exists.

   (i) A sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these values.

   (ii) A monotone sequence that diverges but has a convergent subsequence.

   (iii) A sequence that contains subsequences converging to every point in the infinite set $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$

   (iv) An unbounded sequence with a convergent subsequence.

   (v) A sequence (a_n) that has a bounded subsequence but such that (a_n) contains no convergent subsequence.

   (vi) A Cauchy sequence that is not monotone.

   (vii) A monotone sequence that is not Cauchy.

   (viii) An unbounded sequence that contains a subsequence that is Cauchy.

2. Show that if a sequence of real numbers (a_n) has either of the following properties then it is divergent:

   (i) (a_n) has two subsequences that converge to different limits,

   (ii) (a_n) is unbounded.

   Is the converse true, that any divergent sequence is either unbounded or has two subsequences that converge to a different limit?

3. Let (a_n) be a sequence of nonnegative real numbers that converges to a limit l.

   (i) Prove that the sequence $(\sqrt{a_n})$ converges to $\sqrt{l}$.

   (ii) Prove more generally that for any nonnegative rational $\frac{p}{q}$ the sequence $(a_n^{\frac{p}{q}})$ converges to $(l^{\frac{p}{q}})$.

4.

   (i) Let $c > 0$ be a positive real number. Use the Monotone Convergence Theorem to show that the sequence $(c^{\frac{1}{n}})$ is convergent and determine its limit. [*Consider the cases* $0 < c < 1$ *and* $c > 1$ *separately.*]

   (ii) Is the sequence $(n^{\frac{1}{n}})$ convergent? Either give a proof of divergence or, if it is convergent, determine the limit of the sequence.

5.

(i) Prove that a sequence of reals (a_n) has a monotone subsequence. Deduce that a bounded sequence of reals has a convergent subsequence (Bolzano–Weierstrass Theorem).

(ii) Prove as a corollary of (i) that if a bounded sequence of reals (a_n) has the property that every subsequence that is convergent has the same limit l, then the whole sequence is itself convergent to l.

(iii) Give an example to show that the condition in (ii) that (a_n) is bounded is necessary.

6. Define what it means for a sequence of reals (a_n) to be a *Cauchy sequence*.

(i) Suppose (a_n) is a sequence with the property that for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|a_{n+1} - a_n| < \epsilon$. Is (a_n) necessarily a Cauchy sequence? (Give a counterexample if not, a proof if so.)

(ii) A sequence (a_n) is *contractive* if there is a constant C with $0 < C < 1$ such that

$$|a_{n+2} - a_{n+1}| \leq C|a_{n+1} - a_n|$$

for all $n \in \mathbb{N}$. Prove that a contractive sequence is a Cauchy sequence.

(iii) Show that the sequence (a_n) defined recursively by $a_{n+1} = (2 + a_n)^{-1}$ is contractive when $a_1 > 0$ and determine its limit.