Mathematical Analysis I

Exercise sheet 4

Solutions to selected exercises

29 October 2015

References: Abbott, 2.2, 2.3. Bartle & Sherbert 3.1, 3.2

5. (Cesàro Mean)

(i) Show that if (a_n) is a convergent sequence, then the sequence (b_n) given by the averages

$$b_n = \frac{a_1 + a_2 + \cdots + a_n}{n}$$

also converges to the same limit.

Suppose that $(a_n) \to l$. Then

$$b_n - l = \frac{a_1 + a_2 + \cdots + a_n}{n} - l = \frac{(a_1 - l) + (a_2 - l) + \cdots + (a_n - l)}{n}$$

and so, by definition of convergence of (a_n) to l and the Triangle Inequality, for every $\epsilon > 0$ there is $N \in \mathbb{N}$ such that

$$|b_n - l| \leq \frac{|a_1 - l| + \cdots + |a_N - l| + (n - N)\epsilon}{n} = \epsilon + \frac{|a_1 - l| + \cdots + |a_N - l| - N\epsilon}{n} \tag{1}$$

for all $n \geq N$. For any given $\epsilon > 0, N \in \mathbb{N}$ we can choose $N' \in \mathbb{N}$ such that

$$\left| \frac{|a_1 - l| + \cdots + |a_N - l| - N\epsilon}{n} \right| < \epsilon$$

and then

$$\epsilon + \frac{|a_1 - l| + \cdots + |a_N - l| - N\epsilon}{n} < 2\epsilon$$

and from inequality (1) this proves that $(b_n) \to l$.

(ii) Give an example to show that it is possible for the sequence (b_n) of averages to converge even if (a_n) does not.

One example is $a_n = (-1)^n$, which is a divergent (oscillating) sequence. Here

$$a_1 + a_2 + \cdots + a_n = \begin{cases} -1 & n \text{ even} \\ 0 & n \text{ odd} \end{cases} = \frac{-1 - (-1)^n}{2}$$

and the sequence (b_n) with $b_n = \frac{-1-(-1)^n}{2n}$ converges to 0.

6. Define what it means for a sequence to be *bounded* and for a sequence to be *monotone*.

A sequence (a_n) is bounded if there is $B \in \mathbb{R}$ such that $a_n \in [-B, B]$ for all $n \in \mathbb{N}$.

A sequence (a_n) is monotone increasing if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$ and monotone decreasing if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. A sequence (a_n) is monotone if it is either monotone increasing or monotone decreasing.

(i) Prove that a convergent sequence is bounded. If $(a_n) \to l$ then for every $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $|a_n - l| < \epsilon$ for all $n \geq N$. By the Triangle Inequality,

$$|a_n| \leq |a_n - l| + |l| < \epsilon + |l|$$

for all $n \geq N$, whence

$$|a_n| \leq \max\{|a_1|, \ldots, |a_{N-1}|, |l| + \epsilon\}$$

for all $n \in \mathbb{N}$. This shows (a_n) is bounded.

(ii) Give an example of a bounded sequence that is not convergent. [*This gives a counterexample to the converse of (i).*]

$a_n = (-1)^n$ defines a bounded sequence (a_n) that diverges.

(iii) Use the fact that a bounded set of reals has a supremum to prove that any bounded monotone sequence converges to a limit. [*This is the Monotone Convergence Theorem for sequences.*]

We prove the statement for a bounded monotone increasing sequence (a_n). The case where (a_n) is decreasing is similar or may be deduced from the increasing sequence $(-a_n)$.

Suppose then the (a_n) is increasing and bounded above by B. By the Axiom of Completeness for $\mathbb{R}$ the supremum $a^* = \sup\{a_n : n \in \mathbb{N}\}$ exists and $a^* \leq B$.

By the definition of the supremum as the *least* upper bound, for any $\epsilon > 0$ the number $a^* - \epsilon$ is not an upper bound for $\{a_n : n \in \mathbb{N}\}$. Hence there is $N \in \mathbb{N}$ such that $a^* - \epsilon < a_N \leq a^*$. Since (a_n) is increasing, it then follows that $a^* - \epsilon < a_n \leq a^*$ for all $n \geq N$. This proves that (a_n) converges and $\lim a_n = a^*$.

(iv) Show that the sequence $\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$ defined recursively by $a_{n+1} = \sqrt{2 + a_n}$ is bounded above by 2 and that it is increasing. Deduce from (iii) that (a_n) is convergent and find its limit.

The sequence (a_n) is defined recursively by $a_{n+1} = \sqrt{2 + a_n}$, where $a_1 = \sqrt{2}$.

By induction $a_n < 2$ for all n (base case $a_1 = \sqrt{2} < 2$, induction step $a_{n+1} = \sqrt{2 + a_n} < \sqrt{2+2} = 2$). Also $a_n > 0$ for all $n \in \mathbb{N}$.

Since

$$a_{n+1}^2 - a_n^2 = 2 + a_n - a_n^2 = (2 - a_n)(a_n + 1)$$

and both $a_n + 1 > 0$ and $2 - a_n > 0$ we have $a_{n+1}^2 > a_n^2$, whence $a_{n+1} > a_n$. Thus (a_n) is increasing.

By the Monotone Convergence Theorem (iii) the bounded monotone sequence (a_n) converges to a limit l.

By the algebra of limits applied to the equality $a_n^2 = 2 + a_n$ we have $l^2 = 2 + l$, whence $(l-2)(l+1) = 0$. Since $a_n > -1$ and (a_n) is increasing it follows that $l = 2$.

[*Remark: an explicit – rather than recursive – formula for this sequence is* $a_n = 2\cos\frac{\pi}{2^{n+1}}$. *If you can recall the double angle formula for the cosine function you might see it. It becomes intuitively clear that* $a_n \to 2$ *since* $\cos 0 = 1$ *and* $\theta_n = \pi/2^{n+1} \to 0$ *– later you will prove this intuition correct, specifically because the cosine is a* continuous *function so that* $\lim(\cos\theta_n) = \cos(\lim\theta_n)$ *for a convergent sequence* (θ_n).]