Mathematical Analysis I

Exercise sheet 4

29 October 2015

References: Abbott, 2.2, 2.3. Bartle & Sherbert 3.1, 3.2

1. Define what it means for a sequence (a_n) of reals to *converge* to a limit $l \in \mathbb{R}$.

Use question 6(ii) of Exercise Sheet 2 to explain why the sequence $(\frac{1}{n})$ is convergent with limit 0.

Determine which of the following sequences are convergent and for those which are convergent state (without proof) what the limit is:

(i) $a_n = \frac{an+c}{bn+d}$, where $a, b, c, d \in \mathbb{R}$ and $b \neq 0$.

(ii) $a_n = (-1)^n$

(iii) $a_n = 1 + \frac{(-1)^n}{n}$

(iv) $a_n = 1$ if n is prime and $a_n = 0$ otherwise.

(v) $a_n = \frac{1}{n}$ if n is prime and $a_n = 0$ otherwise.

(vi) $a_n = 1$ if both n and $n+2$ are prime, and $a_n = 0$ otherwise. [*For this sequence you might like to look up the status of the Twin Prime Conjecture.*]

2. Prove that if (a_n) and (b_n) are convergent sequences with limits a and b respectively, then

(i) $(a_n + b_n)$ is convergent and $\lim(a_n + b_n) = a + b$,

(ii) $(a_n b_n)$ is convergent and $\lim(a_n b_n) = ab$,

3. Define what is meant by a *subsequence* of a sequence (a_n).

(i) Prove that if (a_n) is convergent then any subsequence of (a_n) is convergent to the same limit.

(ii) (Squeeze Theorem) Prove that if $a_n \leq c_n \leq b_n$ for all n and $\lim a_n = l = \lim b_n$ then (c_n) is convergent and $\lim c_n = l$.

(iii) Suppose (c_n) is the sequence defined by from the two sequences (a_n) and (b_n) by setting

$$c_n = \begin{cases} a_n & \text{if } n \text{ is odd} \\ b_n & \text{if } n \text{ is even.} \end{cases}$$

Show that if (a_n) and (b_n) are convergent to the same limit l then (c_n) is also convergent with limit l. Does the converse necessarily hold?

4.

(i) Show that if (b_n) is convergent and $b_n \geq 0$ for all n then $\lim b_n \geq 0$.

(ii) Show that if (a_n) is a sequence of positive real numbers such that (a_{n+1}/a_n) is convergent and $\lim \frac{a_{n+1}}{a_n} < 1$ then (a_n) is convergent with limit 0. [*You may assume that when* $0 \leq r < 1$ *the sequence* (r^n) *is convergent with limit* 0.]

(iii) Deduce from (ii) that the sequence $(\frac{n}{2^n})$ converges to 0.

5. (Cesàro Mean)

(i) Show that if (a_n) is a convergent sequence, then the sequence (b_n) given by the averages

$$b_n = \frac{a_1 + a_2 + \cdots + a_n}{n}$$

also converges to the same limit.

(ii) Give an example to show that it is possible for the sequence (b_n) of averages to converge even if (a_n) does not.

6. Define what it means for a sequence to be *bounded* and for a sequence to be *monotone*.

(i) Prove that a convergent sequence is bounded.

(ii) Give an example of a bounded sequence that is not convergent. [*This gives a counterexample to the converse of (i).*]

(iii) Use the fact that a bounded set of reals has a supremum to prove that any bounded monotone sequence converges to a limit. [*This is the Monotone Convergence Theorem for sequences.*]

(iv) Show that the sequence $\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$ defined recursively by $a_{n+1} = \sqrt{2 + a_n}$ is bounded above by 2 and that it is increasing. Deduce from (iii) that (a_n) is convergent and find its limit.