Mathematical Analysis I

Exercise sheet 10

17 December 2015

References: Abbott 5.2, 5.3. Bartle & Sherbert 6.2, 6.3

1. State the Mean Value Theorem.

   (i) By applying the Mean Value Theorem to the function $f(x) = \ln(1+x) - x$ on the interval $[0, x]$ prove that $\ln(1+x) < x$ for $x > 0$. In a similar way, prove that $x - \frac{x^2}{2} < \ln(1+x)$ when $x > 0$.

   Prove the following inequalities by applying the Mean Value Theorem to a suitably defined function and interval:

   (ii) $-x \leq \sin x \leq x$ for $x \geq 0$,

   (iii) $x < \tan x$ for $0 < x < \frac{\pi}{2}$,

   (iv) $\cos x > 1 - \frac{x^2}{2}$ for $x > 0$,

   (v) $e^x > 1 + x + \frac{x^2}{2}$ for $x > 0$,

   (vi) $e^x > 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}$ for $x > 0$. [*For parts (v)-(vi) use only that e^x has derivative e^x (i.e. do not assume the series expansion for* $\exp(x) = e^x$*).*]

2. Let $a < b \in \mathbb{R}$. Suppose $f : [a, b] \to \mathbb{R}$ is continuous and differentiable on (a, b).

   (i) Show that if $f'(x) = 0$ for all $x \in (a, b)$ then f is constant on $[a, b]$.

   (ii) Show that if $f'(x) = A$ for all $x \in (a, b)$ then $f(x) = Ax + B$ for some constants A, B. [*Consider the function $g(x) = f(x) - Ax$.*]

   (iii) Deduce from (i) and (ii) that if $f : [a, b] \to \mathbb{R}$ is twice differentiable and $f''(x) = 0$ on (a, b) then $f(x)$ is a linear function (i.e., $f(x) = Ax + B$ for constants A, B.)

   (iv) Let n be a positive integer. Prove that if f is n times differentiable and $f^{(n)}(x) = 0$ on (a, b), then $f(x)$ is a polynomial of degree $n - 1$. [*Previous parts show this is true for $n = 1, 2$. Induction...*]

3. Use the appropriate version of L'Hospital's Rule to evaluate the following limits:

   (i) $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$,

   (ii) $\lim_{x \to 1} \frac{\ln x}{x - 1}$,

   (iii) $\lim_{x \to \infty} e^{-x} x^n$ (for any fixed positive integer n)

   (iv) $\lim_{x \to \infty} (1 + \frac{1}{x})^x$.

4.

(i) A *fixed point* of a function $f : \mathbb{R} \to \mathbb{R}$ is a value x where $f(x) = x$. Show that if f is differentiable on an interval with $f'(x) \neq 1$ then f can have at most one fixed point.

(ii) A function $f : A \to \mathbb{R}$ is *Lipschitz on A* if there exists an $M > 0$ such that

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq M$$

for all $x, y \in A$. [*There is a uniform bound M on the magnitude of the slopes of lines drawn through any two points on the graph of f.*]

Show that if f is differentiable on a closed interval $[a, b]$ and if f' is continuous on $[a, b]$, then f is Lipschitz on $[a, b]$.

(iii) A function $f : [a, b] \to \mathbb{R}$ is *contractive* if there is a constant $0 < C < 1$ such that

$$|f(x) - f(y)| \leq C|x - y|$$

for all $x, y \in [a, b]$. [*Recall from Sheet 8, question 6, that a contractive function is continuous.*]

Show that if f is continuously differentiable (i.e., f' is continuous) and satisfies $|f'(x)| < 1$ on $[a, b]$ then f is contractive.