Úlohy ke cvičení

Úloha 1: Solve the following systems over the field of complex numbers $\mathbb{C}$.

a) $$\begin{array}{rcrcrcl} 2x_1 & + & (2+2i)x_2 & + & 2ix_3 & = & 1 \\ (1-i)x_1 & + & (1+3i)x_2 & + & (i-1)x_3 & = & 0 \\ (1+i)x_1 & + & (1-i)x_2 & + & (1+i)x_3 & = & 1 \end{array}$$

$$\left(\begin{array}{ccc|c} 2 & 2+2i & 2i & 1 \\ 1-i & 1+3i & i-1 & 0 \\ 1+i & 1-i & 1+i & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 2 & 2+2i & 2i & 1 \\ 2 & 2+2i & 2i & 1 \\ 2 & -2i & 2 & 1-i \end{array}\right) \sim$$
$$\left(\begin{array}{ccc|c} 2 & -2i & 2 & 1-i \\ 0 & 2+4i & 2i-2 & i \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 2 & -2i & 2 & 1-i \\ 0 & 10 & 2+6i & 2+i \\ 0 & 0 & 0 & 0 \end{array}\right)$$

By the backward substitution we get $\mathbf{x} = (\frac{4-3i}{10}, \frac{2+i}{10}, 0)^T + p(\frac{-4-2i}{10}, \frac{-2-6i}{10}, 1)^T$.

For $p = 1$ *and a substitution* $p' = -\frac{p}{5}$ *we get* $\mathbf{x} = (-i/2, -i/2, 1)^T + p'(2+i, 1+3i, -5)^T$.

Úloha 2: Solve the following system of equations over $\mathbb{Z}_5, \mathbb{Z}_7$ and $\mathbb{R}$.

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & 4x_3 & = & 3 \\ 3x_1 & + & x_2 & + & 2x_3 & = & 4 \\ 2x_1 & + & 4x_2 & + & x_3 & = & 3 \end{array}$$

Over $\mathbb{Z}_5$: $\left(\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 3 & 1 & 2 & 4 \\ 2 & 4 & 1 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$ *hence* $\mathbf{x} = (2, 0, 4)^T + (3, 1, 0)^T p$.

Over $\mathbb{Z}_7$: $\left(\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 3 & 1 & 2 & 4 \\ 2 & 4 & 1 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 0 & 4 \end{array}\right)$, *i.e. the system has no solution.*

Over $\mathbb{R}$: $\left(\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 3 & 1 & 2 & 4 \\ 2 & 4 & 1 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 7 & 3 \end{array}\right)$, *hence* $\mathbf{x} = (1, 1/7, 3/7)^T$.

Úloha 3: For $n \in \mathbb{N}$ and an asociative operation $\cdot$ let $a^n = a \cdot a \cdot \ldots \cdot a$, where the element a appears n times in the product.

a) Determine values $2^{101}, 3^{1\,001}$ and $4^{1\,000\,001}$ in the field $\mathbb{Z}_{17}$.

Calculate powers of the base and observe when you get the unit.

$2^8 = 1$, *hence* $2^{101} = 2^{8 \cdot 12 + 5} = (2^8)^{12} \cdot 2^5 = 1^{12} \cdot 15 = 15$.

$3^{16} = 1$ *hence* $3^{1001} = 3^{16 \cdot 62 + 9} = (3^{16})^{62} \cdot 3^9 = 1^{62} \cdot 14 = 14$.

$4^4 = 1$ *hence* $4^{1000001} = 4^{4 \cdot 250000 + 1} = (4^4)^{250000} \cdot 4^1 = 1^{250000} \cdot 4 = 4$.

Úloha 4: Invert the following matrices over fields $\mathbb{Z}_3$ and $\mathbb{Z}_5$

a) $\mathbf{C} = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}$.

Over $\mathbb{Z}_3$: $\mathbf{C}^{-1} = \begin{pmatrix} 0 & 2 & 2 & 1 \\ 2 & 0 & 1 & 0 \\ 1 & 2 & 2 & 1 \\ 2 & 2 & 2 & 2 \end{pmatrix}$, *over* $\mathbb{Z}_5$: $\mathbf{C}^{-1} = \begin{pmatrix} 4 & 3 & 4 & 2 \\ 4 & 0 & 1 & 0 \\ 3 & 4 & 2 & 1 \\ 3 & 3 & 4 & 3 \end{pmatrix}$.

b) $\mathbf{D} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix}$.

$\mathbf{D}$ *is singular in both fields.*