Graph invariants, homomorphisms, and the Tutte polynomial

A. Goodall. J. Nešetřil

January 14, 2013

4 Graph invariants and graph homomorphism profiles

Many graph invariants can be expressed in terms of counting homomorphisms, including the chromatic polynomial (the familiar example, $P(G;k) = \hom(G, K_k)$ for $k \in \mathbb{N}$), the flow polynomial (not so obvious, but we saw how earlier in the course), the Tutte polynomial (also not so obvious [4]), and other polynomial invariants such as the characteristic polynomial.

One of the fundamental questions about a graph invariant is whether it determines a given graph G up to isomorphism: for example, is G determined by its Tutte polynomial, or even just by its chromatic polynomial? What about say the chromatic polynomial and characteristic polynomial jointly: do they together determine G? By using the language of graph homomorphisms we can unify these sorts of question by using *homomorphism profiles.*

4.1 Graph invariants

Let $\mathcal{G}$ denote the set of all finite (multi)graphs up to isomorphism (i.e., graphs in $\mathcal{G}$ are pairwise non-isomorphic, any given graph is isomorphic to exactly one graph in $\mathcal{G}$).

Definition 1. *Let $\mathcal{P} \subseteq \mathcal{G}$ be given in some fixed enumeration $\mathcal{P} = \{P_1, P_2, \ldots\}$. The* left $\mathcal{P}$-profile *of a graph G is the sequence $(\hom(P, G) : P \in \mathcal{P})$ and the* right $\mathcal{P}$-profile *is the sequence $(\hom(G, P) : P \in \mathcal{P})$.*

Definition 2. *A* graph invariant *is a function $f : \mathcal{G} \to S$, where S is a set (often with some algebraic or combinatorial structure that "encodes" some of the graphical combinatorial structure).*

For example, the Tutte polynomial $T(G; x, y)$ is a graph invariant taking values in the ring $\mathbb{Z}[x, y]$. Multiplication in the ring corresponds to the disjoint union of graphs, $T(G_1 \cup G_2; x, y) = T(G_1; x, y)T(G_2; x, y)$. As we have seen, many combinatorial parameters of a graph G are reflected in properties of the

Tutte polynomial $T(G;x,y)$. For example, a graph G with at least two edges is 2-connected if and only if the coefficient of x is non-zero, and G is k-colourable if and only if $T(G;1-k,0)\neq 0$.

The left- (or right-) $\mathcal{P}$-profile defines an invariant taking values in $\mathbb{N}^\omega$, the set of infinite sequences of natural numbers. Multiplication in the monoid $\mathbb{N}^\omega$ corresponds to the disjoint union of graphs for the left-profile, $\hom(G_1\cup G_2,H)=\hom(G_1,H)\hom(G_2,H)$, and to the direct product of graphs for the right-profile, $\hom(F,G_1\times G_2)=\hom(F,G_1)\hom(F,G_2)$.

A graph invariant induces a partition of $\mathcal{G}$ on whose subsets the function f is constant, i.e., two graphs G and G' are *f-equivalent* if $f(G)=f(G')$. If on the other hand $f(G)\neq f(G')$ then the graphs G and G' are *distinguished by f*, belonging as they do to different subsets of the partition of $\mathcal{G}$ induced by f.

If f induces the trivial partition consisting entirely of singletons, then f determines graphs up to isomorphism. There is great interest in finding graph invariants with this property, because of the possible implications for the status of the graph isomorphism problem (still of unknown complexity).

A slightly weaker requirement than that f determine all graphs up to isomorphism is that f determine *almost all* graphs up to isomorphism. Letting $\mathcal{G}(n)\subset\mathcal{G}$ denote the set of all graphs on n vertices, this is to say that

$$\frac{\#\{G\in\mathcal{G}(n),\ G \text{ determined by } f\}}{|\mathcal{G}(n)|}\to 1 \qquad \text{as } n\to\infty.$$

If $\mathcal{G}_1\subset\mathcal{G}$ is a block, or union of blocks, of the partition of $\mathcal{G}$ induced by f then we say the the class $\mathcal{G}_1$ is *determined by f*. In this situation, knowing the value of $f(G)$ we can determine whether $G\in\mathcal{G}_1$. Another way of phrasing this is to say that the property of a graph belonging to the class $\mathcal{G}_1$ is an "f-invariant". For example, the property of being 2-connected is a Tutte polynomial invariant. When $\mathcal{G}_1$ consists of just a single graph G, the graph G itself is determined by f up to isomorphism.

Question 1

(i) Explain why the property of having no cycles is a chromatic polynomial invariant.

(ii) Prove that the complete graph K_k and cycle C_k are both determined by their chromatic polynomials.

Conjecture 3. [2] *Almost all graphs are determined by their chromatic polynomial.*

Bollobás, Pebody and Riordan also make the weaker conjecture – but still far from being solved – that almost all graphs are determined by their Tutte polynomial.

The Tutte polynomial of any forest on m edges is equal to x^m; conversely if $T(G;x,y)=x^m$ then G is a forest on m edges. (Why?) Thus, although forests not individually determined by the Tutte polynomial, the class of all forests on m edges is so determined. (Likewise for the chromatic polynomial, except now one needs to take into account the number of connected components too.)

4.2 Homomorphism profiles determining graph invariants

There are graph invariants that are known to determine each graph G up to isomorphism. An example, trivial by definition, is the equivalence class of the adjacency matrix of G (up to permutation of rows and columns). But despite this triviality one shouldn't overlook the fact that algebraic properties of the adjacency matrix A of a graph G correspond to graphical properties of G in a way that may permit analysis of the latter (for example, the matrix powers of A enumerate walks on G – see below).

The homomorphism $\mathcal{G}$-profile of G, an infinite sequence of natural numbers, may also seem to be too unwieldy a graph invariant to be useful (even allowing that for given G it is possible to truncate the profile to those graphs with at most as many vertices as G). However, we saw in the final lecture how the correspondence $\hom(G, H_1 \times H_2) = \hom(G, H_1)\hom(G, H_2)$ between the direct product of graphs and multiplication in $\mathbb{N}$ could be used to prove the non-trivial result that $G \times G \cong H \times H$ implies $G \cong H$. This required the fact that these profiles do indeed determine all graphs up to isomorphism:

Theorem 4. (Lovász,[5], and [6]) *Let $\mathcal{G}$ be the set of all finite graphs in some enumeration, no two graphs isomorphic.*
Then

(i) The left-$\mathcal{G}$-profile of a (possibly edge-weighted) graph G determines G up to isomorphism.

(ii) The right $\mathcal{G}$-profile of a graph G determines G up to isomorphism.

Can we "thin out" the class $\mathcal{G}$ to make a smaller set $\mathcal{P}$ with the property that every graph is still determined by its left- (and right-) $\mathcal{P}$-profile?

Dvořák [3] has given two examples for left-profiles. A graph H is *k-degenerate* if each subgraph of H contains a vertex of degree at most k. Every graph with tree-width k is k-degenerated. 1-degenerated graphs are precisely forests, but there are 2-degenerated graphs with arbitrary tree-width; the complete graph with each edge subdivided by two new vertices is 2-degenerate.

Theorem 5. (Dvořák, [3]) *Every graph is determined by its left $\mathcal{P}$-profile when*

(i) $\mathcal{P}$ is the set of all 2-degenerate graphs.

(ii) $\mathcal{P}$ consists of all graphs homomorphic to a fixed non-bipartite graph (in other words, an down-set in the homomorphism order with minimal element a non-bipartite graph).

We may extend the terminology of right $\mathcal{P}$-profiles to the case where $\mathcal{P}$ is a collection of edge-weighted graphs.

Question 2 Show the following:

(i) The right $\{K_k : k = 1, 2, \ldots\}$-profile of G determines $P(G; x)$.

(ii) The right $\{K_k^{1-k} : k = 1, 2, \ldots\}$-profile of G determines $F(G; x)$, where K_k^y denotes the complete graph on k vertices with a loop of weight y on each vertex (here $y = 1 - k$).

(iii) The right $\{K_k^y : k, y = 1, 2, \ldots\}$-profile of G determines $T(G; x, y)$. (Here all that matters is that y ranges over some infinite set of values.) [*More fiddly as requires bivariate polynomial interpolation. See [4] for details.*]

(iv) The right $\{K_1^1 + \overline{K}_k : k = 1, 2, \ldots\}$-profile determines the independence polynomial $I(G; x) = \sum x^{|U|}$, where the sum is over all stable sets U in G. (The graph K_1^1 is a single vertex with a loop attached; the graph $K_1^1 + \overline{K}_k$ the star $K_{1,k}$ with a loop on its central vertex.)

Conjecture 3 thus states that the right $\{K_k : k = 1, 2, \ldots\}$-profile determines almost all graphs (or in its weaker form, that the right $\{K_k^y : k, y = 1, 2, \ldots\}$-profile determines almost all graphs).

How about the right $\{K_1^1 + \overline{K}_k : k = 1, 2, \ldots\}$-profile? Well, as Noy showed [7], using the fact that on average a random graph on n vertices has independence (stability) number $O(\log n)$, almost all graphs are *not* determined by the independence polynomial. So here we have an example of a homomorphism profile by an infinite number of non-isomorphic graphs for which we know it is not true that the profile determines almost all graphs.

4.3 Spectrum and degree sequence by left profiles

A *k-walk* in a graph is an alternating sequence of vertices and edges $v_0, e_1, v_1, e_2, \ldots, v_{k-1}, e_k, v_k$, where $e_{i+1} = v_i v_{i+1}$ for $0 \leq i \leq k-1$. A k-walk is *closed* if $v_0 = v_k$. A 0-walk is just a vertex and is always closed. A 1-walk is a walk from a vertex to an adjacent vertex. A closed 1-walk is a loop.

Lemma 6. *Let H be an edge-weighted graph with adjacency matrix A. Then*

$$\hom(C_k, H) = \operatorname{tr}(A^k).$$

Proof. The matrix A^k has (i,j) entry the sum of edge-weighted k-walks from i to j, as can be proved by induction. (The weight of a walk is the product of its edge weights, with multiplicities counted for repeated edges.) A closed k-walk corresponds to a homomorphic image of C_k. The diagonal entries of A^k then together sum to $\hom(C_k, H)$. By diagonalization, $A = B^{-1}DB$ for orthogonal matrix B and diagonal matrix $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, where the λ_i are the eigenvalues of A taken with multiplicity. ☐

Corollary 7. *Let H be an edge-weighted graph H on n vertices with adjacency matrix A. Then the left $\{C_k : 1 \leq k \leq n\} \cup \{K_1\}$-profile of H determines the spectrum of A.*

Proof. If A has eigenvalues $\lambda_1, \ldots, \lambda_n$ then $\mathrm{tr}(A^k) = \sum_i \lambda_i^k$. In particular, $\mathrm{tr}(A^0) = \hom(K_1, H) = n$ gives the number n of vertices of H, i.e., the size of A. Given the power sums $\sum \lambda_i^k$ for $1 \leq k \leq n$, Newton's relations yield the elementary symmetric polynomials in the λ_i and hence the λ_i are uniquely determined (as the roots of the characteristic polynomial of A). $\square$

Restricting attention to simple unweighted undirected graphs, graphs determined by their spectrum include K_n, $K_{n,n}$ and C_n. (Curiously, the line graphs $L(K_n)$ of complete graphs are also determined by their spectrum with the exception of the case $n = 8$, where there are three other non-isomorphic graphs with the same spectrum.) Similar to Conjecture 3 about the chromatic polynomial, it is conjectured that almost all graphs are determined by their spectrum [8]. On the other hand, almost all trees are not determined by their spectrum, and there are many constructions of cospectral non-isomorphic graphs. The smallest pair of graphs with the same spectrum is $C_4 \cup K_1$ and $K_{1,4}$.

The *characteristic polynomial* of G is defined by $\phi(G; x) = \det(A - xI) = (x - \lambda_1) \cdots (x - \lambda_n)$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A (taken with multiplicity). So in all the above we could have talked about the characteristic polynomial of G (rather than the spectrum) being determined by the left $\{K_{1,k} : k = 0, 1, , \ldots\}$-profile of G.

By Corollary 7 the conjecture of Van Dam and Haemers [8] is that almost all graphs are determined by their left $\{K_{1,k} : k = 0, 1, , \ldots\}$-profile.

Lemma 8. *Let H be an edge-weighted graph on n vertices with adjacency matrix A, and let $\mathbf{1}$ denote the $n \times 1$ all-one vector. Then the left $\{K_{1,k} : 1 \leq k \leq n\}$-profile of H determines the vector $A\mathbf{1}$.*

Proof. The homomorphic image of $K_{1,k}$ is a multiset of k edges incident with a common vertex. If H has vertex set $[n] = \{1, \ldots, n\}$ and adjacency matrix $A = (a_{u,v})_{u,v \in [n]}$ then

$$\hom(K_{1,k}, H) = \sum_{v \in [n]} \Big(\sum_{u \in [n]} a_{u,v} \Big)^k,$$

by taking all possible choices of a multiset of k edges incident with common vertex v as the image of $K_{1,k}$. By taking $k = 1, \ldots, n$ we can determine the column sums $\sum_{u \in [n]} a_{u,v}$ of A, i.e., the vector $\mathbf{1}^\top A$. Since A is symmetric this also gives the row sums and the vector $A\mathbf{1}$. $\square$

When H is an unweighted graph, i.e. its adjacency matrix has entries either 0 or 1, with degree sequence $d_1, \ldots, d_n$ (vertex degrees listed in non-increasing order), we have as a particular case of Lemma 8 that

$$\hom(K_{1,k}, H) = \sum_i d_i^k.$$

(The case $k = 0$ gives us n: $\hom(K_1, H) = |V(H)|$.) By taking $k = 0, 1, \ldots n$ we obtain the following:

Corollary 9. *The left* $\{K_{1,k} : k = 0, 1, \ldots\}$*-profile of (an unweighted graph)* H *determines the degree sequence of* H.

Are almost all graphs determined up to isomorphism by their degree sequence? If ab and cd are edges of a graph then the (multi)graph obtained from G by deleting ab and cd and replacing these by edges ac and bd has the same degree sequence: provided ac and bd are not already edges then this gives another simple graph G' with the same degree sequence as G. Of course, it may be that there are no such pair of edges ab and cd for which this exchange both yields a simple graph and one that is non-isomorphic to G'.

References

[1] N. Biggs, *Algebraic Graph Theory*, 2nd ed., Cambridge Univ. Press, Cambridge, 1993

[2] B. Bollobás, L. Pebody, O. Riordan, Contraction-deletion invariants for graphs, J. Combin. Theory Ser. B 80 (2000) 320–345.

[3] Z. Dvořák, On recognizing graphs by number of homomorphisms, J. Graph Theory, 64:4 (2010), 330–342

[4] D. Garijo, A.J. Goodall, and J. Nešetřil, Distinguishing graphs by left and right homomorphism profiles, European J. Combin. 32 (2011), 1025–1053

[5] L. Lovász, Operations with structures, Acta Math. Hung. 18 (1967) 321–328.

[6] L. Lovász, The rank of connection matrices and the dimension of graph algebras, European J. Combin. 27 (2006) 962–970

[7] M. Noy, Graphs determined by polynomial invariants, Theoret. Comput. Sci. 307:2 (2003), 365–384

[8] E.R. van Dam, W.H. Haemers. Which graphs are determined by their spectrum? Linear Algebra Appl. 373 (2003), 241–272