Discrete Mathematics

Exercise sheet 9

28 November/ 6 December 2016

1. How many graphs on the vertex set $[2n] = \{1, 2, \ldots, 2n\}$ are isomorphic to the graph consisting of n vertex-disjoint edges (i.e. with edge set $\{\{1, 2\}, \{3, 4\}, \ldots, \{2n-1, 2n\}\}$?

Any such graph arises by pairing off the $2n$ vertices that are to be joined by edges. The number of ways to do this can be counted as follows: choose which vertex to pair with vertex 1 ($2n - 1$ choices). This leaves $2n-2$ vertices to pair off. Choose which vertex to pair off with the smallest vertex remaining ($2n - 3$ choices). Repeat this procedure at step $i \in [n]$ by taking the smallest of the remaining vertices and deciding which one it will pair off with. At step i there are $2n - 2i + 1$ free choices of which vertex to pair off with the smallest remaining vertex. Multiplying these together we find there are

$$(2n - 1)(2n - 3)(2n - 5) \cdots 3 \cdot 1 = (2n - 1)!!$$

ways in total, and this is the number of graphs on $[2n]$ isomorphic to the given graph consisting of n vertex-disjoint edges.

The proof can be formalized by induction.

Base case $n = 1$: There is $1 = 1!!$ graph consisting of a single edge joining 2 vertices.

Induction hypothesis: there are $(2n–1)(2n–3) \cdots 3 \cdot 1$ graphs on $[2n]$ isomorphic to the graph consisting of n vertex-disjoint edges.

Induction step: A graph on $[2(n+1)]$ is isomorphic to the graph consisting of $n + 1$ vertex-disjoint edges if and only if it has one isolated edge $\{1, i\}$ whose removal leaves a graph on $[2(n + 1)] \setminus \{1, i\}$ isomorphic to the graph consisting of n vertex-disjoint edges. There are $2n + 1$ choices for i and by hypothesis $(2n-1)(2n-3) \cdots 3 \cdot 1$ graphs on $[2(n + 1)] \setminus \{1, i\}$ that are isomorphic to a graph consisting of n vertex-disjoint edges.

Hence there are $(2n+1) \cdot (2n–1)(2n–3) \cdots 3 \cdot 1$ graphs on $[2(n+1)]$ isomorphic to the graph consisting of $n + 1$ vertex-disjoint edges. $\square$

Remark The number of ordered set partitions[1] of a set of size m into r subsets, of sizes $k_1, \ldots, k_r$, is the multinomial coefficient

$$\frac{m!}{k_1! \cdots k_r!},$$

which can be derived by repeated application of the formula for the number of combinations of m things taken k at a time,

$$\binom{m}{k} = \frac{m!}{k!(m-k)!}.$$

By choosing $A_1 \subseteq A$, $A_2 \subseteq A \backslash A_1, \ldots, A_r \subseteq A \backslash (A_1 \cup \cdots \cup A_{r-1})$ in orderm the number of ordered set partitions $(A_1, \ldots, A_r)$ with $|A_i| = k_i$ is

$$\frac{m!}{k_1! \cdots k_r!} = \binom{m}{k_1}\binom{m-k_1}{k_2}\binom{m-k_1-k_2}{k_3} \cdots \binom{m-k_1-\cdots-k_{r-1}}{k_r}.$$

[1] An *ordered set partition* of a set A is a sequence of subsets $(A_1, \ldots, A_r)$ such that $A_1 \cup A_2 \cup \cdots A_r = A$ and the sets $A_1, \ldots, A_r$ are pairwise disjoint. A(n unordered) *set partition* of A is a set of subsets $\{A_1, \ldots, A_r\}$ such that $A_1 \cup A_2 \cup \cdots A_r = A$ and the sets $A_1, \ldots, A_r$ are pairwise disjoint. To each set partition into r non-empty sets there corresponds $r!$ ordered set partitions, obtained by taking all the different possible orderings of the subsets in the partition. When there are s empty subsets, we need to adjust by a factor of $s!$ to account for permuting the empty subsets among themselves.

If $k_i \geq 1$ for each i then the number of *unordered* set partitions of m elements into subsets of sizes $k_1, \ldots, k_r$ is given by

$$\frac{m!}{r! k_1! \cdots k_r!}.$$

In particular, the number of (unordered) set partitions of $2n$ vertices into n subsets of size 2 (corresponding to edges) is given by

$$\begin{aligned}
\frac{(2n)!}{n!(2!)^n} &= \frac{(2n) \cdot 2(n-1) \cdots 4 \cdot 2 \ \cdot \ (2n-1)(2n-3) \cdots 3 \cdot 1}{2^n n!} \\
&= \frac{(2n) \cdot 2(n-1) \cdots 4 \cdot 2 \ \cdot \ (2n-1)(2n-3) \cdots 3 \cdot 1}{(2n) \cdot 2(n-1) \cdots 4 \cdot 2} \\
&= (2n-1)(2n-3) \cdots 3 \cdot 1
\end{aligned}$$

2. Let G be a graph with adjacency matrix A_G. Show that G contains a triangle (i.e. a copy of K_3) if and only if there exist indices i and j such that both the matrices A_G and A_G^2 have a nonzero entry in the (i, j)-position.

We may assume $i \neq j$: if $i = j$ then $(A_G)_{i,i} = 0$ as loops are not allowed. On the other hand we have $(A_G^2)_{i,i} = \deg(i)$ since for each vertex k adjacent to i there is the closed walk i, ik, k, ik, i of length 2.

If G contains a triangle on vertices i, j, k then $(A_G)i, j = 1$ and $(A_G^2)_{i,j} \geq 1$ since i, ik, k, kj, j is a walk of length 2 from i to j.

Conversely, if $(A_G)_{i,j} \neq 0$ and $(A_G^2)_{i,j} \neq 0$ then, by definition of the adjacency matrix, $(A_G)_{i,j} = 1$ and there is an edge ij, and $(A_G^2)_{i,j} \geq 1$ so there is at least one walk from i to j of length 2. Let this walk be i, ik, k, kj, j for a vertex k. Then $k \notin \{i, j\}$ since there are no loops. We then have a triangle, traversed by the closed walk i, ik, k, kj, j, ij, i.

3. Let G be a graph with 9 vertices, each of degree 5 or 6. Prove that it has at least 5 vertices of degree 6 or at least 6 vertices of degree 5.

We have $p_5 + p_6 = 9$ and $5p_5 + 6p_6$ equal to twice the number of edges is even. The number $5p_5 + 6p_6$ is even if and only if p_5 is even.

Suppose $p_5 \leq 5$ and $p_6 \leq 4$. Then $p_5 = 5$ and $p_6 = 4$ by the vertex count $p_5 + p_6 = 9$, but p_5 must be even, a contradiction.

Hence $p_5 \geq 6$ or $p_6 \geq 5$.

4. Let T be a tree with n vertices, $n \geq 2$. For a positive integer i, let p_i be the number of vertices of T of degree i.

(a) Prove that

$$p_1 - p_3 - 2p_4 - \cdots - (n-3)p_{n-1} = 2.$$

First note that $p_i = 0$ for $i \geq n$ (a vertex can have degree at most $n - 1$, which happens for the star $K_{1,n-1}$).

Using the fact that T has n vertices and $n - 1$ edges, we have

$$p_1 + p_2 + p_3 + \cdots + p_{n-1} = n$$

and

$$p_1 + 2p_2 + 3p_3 + \cdots + (n-1)p_{n-1} = 2(n-1).$$

Subtracting the second equation from twice the first, we obtain

$$p_1 - 2p_3 - \cdots - (n-3)p_{n-1} = 2. \tag{1}$$

(When $n = 2$ we have $p_1 = 2$.)

(b) Deduce from (a) the end-vertex lemma, that a tree with at least two vertices has at least two end-vertices.

By equation (1), $p_1 = 2 + p_3 + 2p_4 + \cdots + (n-3)p_{n-1}$ When $n \geq 3$ this implies $p_1 \geq 2$ since the p_i are nonnegative integers.

(c) Deduce from (a) that a tree with a vertex of degree k has at least k vertices of degree 1.

By equation (1), when $p_k \geq 1$ we have $p_1 = 2 + p_3 + 2p_4 + \cdots + (n-3)p_{n-1} \geq 2 + (k-2)p_k \geq k$.

Remark Direct proofs of (b) and (c) are as follows:

To show there are at least two endvertices (degree 1), consider a path of maximum length in T. This has length at least 1 and its endpoints are endvertices of T (if not, then we could extend the path to a longer one).

Given a vertex v of degree k, deleting v leaves a forest comprising k trees. By (b) each component contains at least two leaves, one of which was attached to v in T, or consists of an isolated vertex, which is a leaf in the orginal tree T attached to v. This shows that there must be at least k leaves in T.