Discrete Mathematics

Exercise sheet 7

14/ 20 November 2016

1. A restaurant cook had the misfortune of losing her engagement ring in a big cauldron of soup, and the carelessness to lose her wedding ring after it, which also found its way into the soup. The cook was for some reason not too painstaking in searching for her lost rings and served up all the soup until the pot was completely empty. The soup was divided among 25 guests, among whom 8 were women. What is the probability that

The possible outcomes Ω for this problem consist of pairs (person, ring), where "person" ranges over the 25 guests and "ring" ranges over engagement ring and wedding ring, and the elementary event (person, ring) occurs when the given person gets a ring of given type.

The probability of one of the $25^2 = 625$ elementary events is $\frac{1}{625}$ (uniform distribution).

(a) one person got both rings?

The event that one particular person gets both rings occurs with probability $\left(\frac{1}{25}\right)^2$ (independence of events that has engagement ring and has wedding ring in soup) and hence that one of the 25 people do is

$$25 \cdot \left(\frac{1}{25}\right)^2 = \frac{1}{25}.$$

(Alternative argument: the probability that a person gets the wedding ring given that he/she gets the engagement ring is $\frac{1}{25}$, and these events are independent.)

(b) no man got a ring?

Probability that a woman gets engagement ring is $\frac{8}{25}$ and that a woman gets wedding ring is $\frac{8}{25}$. Hence the probability that no man got a ring is

$$\frac{8}{25} \cdot \frac{8}{25} = \frac{64}{625}.$$

(c) two men got a ring each?

There is probability $\frac{17}{25}$ that a man gets the engagement ring, and probability $\frac{16}{25}$ that a different man gets the wedding ring. Hence the probability is

$$\frac{17}{25}\frac{16}{25} = \frac{272}{625}.$$

(d) a man got one ring, a woman the other? The probability of this event is the probability that a man gets the engagement ring and a woman the wedding ring plus the probability that a man gets the wedding ring and a woman the engagement ring:

$$\frac{17}{25} \cdot \frac{8}{25} + \frac{8}{25} \cdot \frac{17}{25} = \frac{272}{625}.$$

2. In this question we assume dice are fair (unbiased), in the sense that each of the six possible scores of a die are equally likely. Additionally, successive throws of a die are independent, the outcome of one throw not affecting that of another.

(a) What is the probability that throwing two dice yields an even total score? A multiple of 3? Space of possible outcomes is $[6] \times [6] = \{(x, y) : 1 \leq x, y \leq 6\}$, the probability of each elementary event being $\frac{1}{36}$.

The event that there is an even score has probability $\frac{1}{2}$ because there are three odd numbers between one and six and three even numbers between one and size, so the probability their sum is even must equal the probability that their sum is odd.[1]

Alternatively, the event there is an even score is given by

$$\{(x, y) \in [6] \times [6] : x + y \equiv 0 \pmod 2\}$$

and enumerating all possiblities $(1, 1), (1, 3), \ldots, (6, 6)$ gives 18 in all and probability $\frac{18}{36} = \frac{1}{2}$.

There are 12 pairs scoring a multiple of 3 ($1+2 = 1+2$, $1+5 = 2+4 = 3+3 = 4+2 = 5+1$, $3 + 6 = 4 + 5 = 5 + 4 = 6 + 3$ and $6 + 6$), so the probability is $\frac{12}{36} = \frac{1}{3}$.

(b) Determine the probability six throws of a die yield a score of three or more at least three times.

A score of at least three on one die has probablity $\frac{4}{6} = \frac{2}{3}$. The probability that exactly i dice among the six have score three or more is then

$$\binom{6}{i}\left(\frac{2}{3}\right)^i\left(\frac{1}{3}\right)^{6-i}.$$

Summing this over $i = 3, 4, 5, 6$ gives

$$\binom{6}{3}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^3+\binom{6}{4}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^2+\binom{6}{5}\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^1+\binom{6}{6}\left(\frac{2}{3}\right)^6\left(\frac{1}{3}\right)^0=\frac{656}{729}.$$

[1] Let (X, Y) be the pair of random variables giving the scores on the first die and second die. Then,

$$\begin{aligned}\mathbb{P}(X + Y \equiv 0 \bmod 2) &= \mathbb{P}(X \equiv 0 \bmod 2)\mathbb{P}(Y \equiv 0 \bmod 2) + \mathbb{P}(X \equiv 1 \bmod 2)\mathbb{P}(Y \equiv 1 \bmod 2)\\ &= \frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}\\ &= \frac{1}{2},\end{aligned}$$

where we have used the fact that X and Y are independent.

Similarly,

$$\begin{aligned}\mathbb{P}(X+Y \equiv 0 \bmod 3) &= \mathbb{P}(X \equiv 0 \bmod 3)\mathbb{P}(Y \equiv 0 \bmod 3) + \mathbb{P}(X \equiv 1 \bmod 3)\mathbb{P}(Y \equiv 2 \bmod 3)) + \mathbb{P}(X \equiv 2 \bmod 3)\mathbb{P}(Y \equiv 1 \bmod 3)\\ &= \frac{1}{3}\cdot\frac{1}{3}+\frac{1}{3}\cdot\frac{1}{3}+\frac{1}{3}\cdot\frac{1}{3}\\ &= \frac{1}{3},\end{aligned}$$

since $3, 6 \equiv 0 \bmod 3$, $1, 4 \equiv 1 \bmod 3$ and $2, 5 \equiv 2 \bmod 3$.

How about the probability the sum is a multiple of 4? We have

$$\mathbb{P}(X \equiv 0 \bmod 4) = \frac{1}{6} = \mathbb{P}(X \equiv 3 \bmod 4), \quad \mathbb{P}(X \equiv 1 \bmod 4) = \frac{2}{6} = \mathbb{P}(X \equiv 2 \bmod 4)$$

so the probability we get a score that is a multiple of 4 is

$$\frac{1}{6}\cdot\frac{1}{6}+\frac{2}{6}\cdot\frac{1}{6}+\frac{1}{6}\cdot\frac{2}{6}+\frac{2}{6}\cdot\frac{2}{6}=\frac{9}{36}=\frac{1}{4}.$$

The previous might make you think that the probability of scoring a sum which is a multiple of 5 will be $\frac{1}{5}$. However, can you see why this probability cannot be $\frac{1}{5}$ without performing the calculation?

(c) What is the probability that in three successive throws the scores are strictly increasing?

Let $(x, y, z) \in [6] \times [6] \times [6]$ be the triple of scores observed in three consecutive throws. The number of triples with $x < y < z$ is equal to $\binom{6}{3}$ (to each such triple (x, y, z) corresponds a subset $\{x, y, z\}$ and, conversely, for each subset $\{x, y, z\} \subseteq [6]$ there is a single way to put them increasing order).

Hence the probability is

$$\binom{6}{3} \cdot \frac{1}{6^3} = \frac{20}{216} = \frac{5}{54}.$$

(d) What is the probability that in three successive throws the scores are nondecreasing?

To an outcome $x \leq y \leq z$ takes one of the following forms: (a, a, a), (a, a, b), (a, b, b), (a, b, c), where $a < b < c$.

The first gives 6 (choose a), the second $\binom{6}{2}$ (choose a and b), the third $\binom{6}{2}$ and the fourth $\binom{6}{3}$. This gives $6 + 15 + 15 + 20 = 56$ in all. Hence the probability is

$$\frac{56}{216} = \frac{7}{27}.$$

(e) Answer (c) and (d) for k successive throws rather than three, where $k \in \mathbb{N}$.

Note that for $k > 6$ throws it is impossible to have strictly increasing values, so we may assume $k \leq 6$. When $k = 6$ there is only one outcome in the event, namely $x_i = i$ for $i = 1, 2, \ldots, 6$.

Let $(x_1, x_2, \ldots, x_k) \in [6]^k$ be the k-tuple of scores observed in k consecutive throws. The number of k-tuples with $x_1 < x_2 < \cdots < x_k$ is equal to $\binom{6}{k}$ (to each such k-tuple $(x_1, x_2, \ldots, x_k)$ corresponds a k-subset $\{x_1, x_2, \ldots, x_k\}$ and, conversely, for each subset $\{x_1, x_2, \ldots, x_k\} \subseteq [6]$ there is a single way to put the elements in increasing order).

Hence the probability is

$$\binom{6}{k} \cdot \frac{1}{6^k}.$$

(Note that this formula holds for $k > 6$ as well, since $\binom{6}{k} = 0$ for $k > 6$.)

For the second event of non-decreasing scores, again let $(x_1, x_2, \ldots, x_k) \in [6]^k$ be the k-tuple of scores observed in k consecutive throws.

To a k-tuple $(x_1, x_2, \ldots, x_k)$ with $x_1 \leq x_2 \leq \cdots \leq x_k$ corresponds a multiset on $[6]$ whose elements are $x_1, x_2, \ldots, x_k$, and, conversely, given such a multiset, listing the elements in non-decreasing order yields such a k-tuple.

Hence there are $\binom{6+k-1}{k} = \binom{5+k}{k}$ such k-tuples and the probability that the k successive scores are non-decreasing is given by

$$\binom{5+k}{k} \cdot \frac{1}{6^k}.$$

Check: when $k = 1$ this is probability 1, corresponding to the fact that a sequence of one element is vacuously non-decreasing, and when $k = 3$ this is $\binom{8}{3} \cdot \frac{1}{6^3}$, as obtained in part (d).

3. (Birthday Paradox) For this problem we ignore leap years and assume that a person has his/her birthday among one of the 365 days of the calendar year.

(a) Show that there is more than a 50% chance that in a room containing 23 people there are two who share a birthday. [*Hint: show that the probability that everyone has different birthdays is less than* $\frac{1}{2}$, *using a calculator/computer to perform the arithmetic.*]

The probability that 23 people have *different* birthdays is

$$\begin{aligned}\left(1-\frac{1}{365}\right)\left(1-\frac{2}{365}\right)\cdots\left(1-\frac{22}{365}\right) &= \frac{365}{365}\cdot\frac{364}{365}\cdots\frac{343}{365}\\ &= \left(\frac{1}{365}\right)^{23}(365\cdot 364\cdot 363\cdots 343)\\ &\approx 0.492703\end{aligned}$$

from which the complementary event that some pair of people share a birthday is $\approx 0.507297 \approx 50.7\%$.

[Remark: the argument that a given pair of people have probability $\frac{364}{365}$ of not having the same birthday, and there are $\binom{23}{2} = 253$ pairs of people, so the probability that none of them share a birthday is $\left(\frac{364}{365}\right)^{253}$ is invalid because the events that pairs of people have different birthdays are not independent. For example, $\mathbb{P}(x = z|x = y, y = z) = 1$ while $\mathbb{P}(x = z) \neq 1$.]

(b) Use the approximation $1 - x \approx e^{-x}$ for small x to show that among n people the chance that two people share a birthday is close to $1 - e^{-n^2/730}$.

The probability that no pair of people share a birthday among n people is given by

$$\begin{aligned}\left(1-\frac{1}{365}\right)\left(1-\frac{2}{365}\right)\cdots\left(1-\frac{n-1}{365}\right) &\approx e^{-1/365}\cdot e^{-2/365}\cdots e^{-(n-1)/365}\\ &= e^{-(1+2+\cdots+(n-1))/365}\\ &= e^{-n(n-1)/(2\cdot 365)}\end{aligned}$$

Hence the probability that some pair of people share a birthday among n people is approximately

$$1 - e^{-n(n-1)/730} \approx e^{-n^2/730}.$$

If n objects are put into d boxes (in the above, a box is a date on the calendar year and $d = 365$) then a similar argument gives the probability that at least two out of the n objects are put in the same box is given approximately by $1 - e^{-n^2/2d}$, provided n/d is small (so as to apply the approximation $e^{-x} \approx 1 - x$). In particular, to obtain probability approximately p we require

$$1 - p \approx e^{-n^2/2d}$$

from which

$$\begin{aligned}\ln(1-p) &\approx -n^2/2d\\ 2d\ln\frac{1}{1-p} &\approx n^2\end{aligned}$$

i.e.

$$n \approx \sqrt{2\ln\frac{1}{1-p}}\cdot\sqrt{d}.$$

(Put $p = \frac{1}{2}$ and $\sqrt{2 \ln 2} \approx 1.177$ so n should be around $1.177\sqrt{d}$ which is approximately 22.5 for $d = 365$, thereby explaining why we take 23 people in stating the above "Birthday Paradox.")

Note. The Pigeonhole Principle states that if we take $n = d + 1$ then we are guaranteed to have some pair of objects in the same box, i.e. the probability is 1 that when $d + 1$ objects are put in d boxes there is some pair of objects put in the same box. To get probability $p = 1 - \epsilon$ requires $n \approx \sqrt{2 \ln \frac{1}{\epsilon}} \cdot \sqrt{d}$. For $\epsilon = \frac{1}{100}$ this is approximately $9.2\sqrt{d}$.