Discrete Mathematics

Exercise sheet 4

24 October/ 1 November 2016

1. [*Bookwork*] Let $R \subseteq X \times X$ be a relation on a set X. Define what is means for R to be

(a) reflexive, $\forall x \in X \quad (x, x) \in R$

(b) symmetric, $\forall x, y \in X \quad (x, y) \in R \Rightarrow (y, x) \in R$

(c) anti-symmetric, $\forall x, y \in X \quad [x \neq y \wedge (x, y) \in R] \Rightarrow (y, x) \notin R.$
Alternatively, $\forall x, y \in X \quad [(x, y) \in R \wedge (y, x) \in R] \Rightarrow x = y.$

(d) transitive, $\forall x, y, z \in X \quad [(x, z) \in R \wedge (z, y) \in R] \Rightarrow (x, y) \in R$

(e) an equivalence relation, reflexive, symmetric and transitive

(f) a partial order, reflexive, anti-symmetric and transitive

(g) a linear order. partial order in which every pair of elements are comparable, i.e., $\forall x, y \in X \quad [(x, y) \in R \vee (y, x) \in R]$.

2. The *adjacency matrix* of a binary relation R on $[n] = \{1, 2, \ldots, n\}$ is the matrix whose (i, j)-entry is defined for $i, j \in [n]$ by

$$a_{i,j} = \begin{cases} 1 & (i, j) \in R \\ 0 & (i, j) \notin R. \end{cases}$$

(See Section 1.5 of Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, for a detailed exposition.)

(a) How many relations are there on $[n]$ in total? [*Hint: an* $n \times n$ *matrix with entries* 0 *or* 1 *defines the adjacency matrix of a relation. Count how many such matrices there are.*]

Think of the matrix as a function from $[n] \times [n]$ to $\{0, 1\}$. There are $2^{|[n] \times [n]|} = 2^{n^2}$ such functions, each specifying a unique relation on $[n] \times [n]$.

(b) How many reflexive relations are there on $[n]$?

For a reflexive relation the adjacency matrix must have $a_{i,i} = 1$ for $i = 1, \ldots, n$. The other entries $a_{i,j}$ with $i \neq j$ can be 0 or 1 independently. Hence there are $2^{n^2 - n}$ reflexive relations on $[n]$.

(c) How many symmetric relations are there on $[n]$?

For a symmetric relation we must have $a_{j,i} = a_{i,j}$ for each $i, j \in [n]$ (the adjacency matrix is equal to its own transpose). Once $a_{i,j}$ has been specified for $i \leq j$, the remaining entries are determined.

Hence there are $2^{\frac{n^2 - n}{2} + n} = 2^{\frac{1}{2} n(n+1)}$ symmetric relations on $[n]$.

(d) How many anti-symmetric relations are there on $[n]$? [*Hint: for a pair (i,i) there are two choices (either $(i,i) \in R$ or $(i,i) \notin R$), while for (i,j) with $i \neq j$ there are three mutually exclusive choices, $(i,j) \in R$, $(j,i) \in R$ or neither.*]

For each entry $a_{i,i}$ there are 2 possibilities, namely 0 or 1, making 2^n in total for these diagonal entries. For each of the $\frac{1}{2}(n^2-n)$ pairs of entries $(a_{i,j}, a_{j,i})$ with $i < j$ there are 3 possibilities, namely $(1,0),(0,1),(0,0)$, making $3^{\frac{1}{2}(n^2-n)}$ in total. Hence there are $2^n \cdot 3^{\binom{n}{2}}$ anti-symmetric relations on $[n]$.

(e) How many linear orders are there on $[n]$? [*You may find the adjacency matrix point of view not so helpful to answer this question, but rather take another viewpoint.*]

A linear order on $[n]$ is determined by a permutation of $[n]$, a function $f : [n] \to [n]$ such that $f(i) \preceq f(j)$ when $i \leq j$. Thus $f(1) \preceq f(2) \preceq f(3) \preceq \cdots \preceq f(n)$. Each of the $n!$ permutations of $[n]$ determines in this way a linear order on $[n]$.

Hence there are $n!$ linear orders on $[n]$.

For any partial order $(X, \preceq)$ (satisfying reflexivity, anti-symmetry and transitivity) there corresponds a *strict partial order* $(X, \prec)$ satisfying *irreflexivity*, anti-symmetry and transitivity: define $x \prec y$ iff $x \preceq y$ and $x \neq y$.

Conversely, given a strict partial order $(X, \prec)$ there corresponds a partial order $(X, \preceq)$ in which $x \preceq y$ iff $x \prec y$ or $x = y$.

The number of linear orders is the same as the number of strict linear orders, by this correspondence.

At the end of the class there was a potential confusion raised which happily is not one after all: in defining a partial order on $[n]$ the elements of $[n]$ are *distinct* and cannot be identified in producing a linear order, without making a linear order on fewer than n elements. Further, in a (strict) linear order, any two distinct elements are comparable, so we have a bijection between permutations of $[n]$ and linear orders, and between permutations of $[n]$ and strict linear orders.

3. Let D_n be the set of divisors of n. Show that the relation $\preceq$ on D_n defined by $a \preceq b$ if and only if a divides b is a partial order.

By definition, for $a, b \in \mathbb{N}$, $a|b$ iff $b = xa$ for some $x \in \mathbb{N}$.

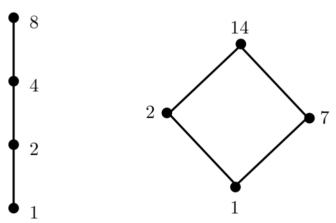
Reflexive: $a|a$ since $a = 1a$.

Anti-symmetric: if $a|b$ and $b|a$ then $b = xa$, $a = yb$ for some $x, y \in \mathbb{N}$, whence $a = yxa$, from which $yx = 1$, and $y = x = 1$. Thus $a = b$.

Transitive: if $a|c$ and $c|b$ then $c = ya, b = xc$ for some $x, y \in \mathbb{N}$, from which $b = xya$, and so $a|b$.

(b) For $n = 2, 3, \ldots, 11$ draw the Hasse diagram of the poset $(D_n, \preceq)$ of divisors of n.

For example, the posets of divisors of 8 and 14 are as below:



(c) What property does the number n have if $(D_n, \preceq)$ is a linear order (as for $n = 8$)?

$n = p^a$ for some prime p and integer $a \geq 1$.

(d) When is $(D_n, \preceq)$ isomorphic to the poset $([m], \subseteq)$ for some m (as is the case for $n = 14$ with $m = 2$)?

When $n = p_1 p_2 \cdots p_m$ for distinct primes $p_1, \ldots, p_m$. (A divisor of n takes the form $p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$ for $a_1, a_2, \ldots, a_m \in \{0, 1\}$. These are in one-to-one correspondence with subsets of $[m]$ by reading a_i as the indicator function of the subset $A \subseteq [m]$ defined by $i \in A$ iff $a_i = 1$.)

(e) What is the size of the longest chain in $(D_n, \preceq)$?

If $n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_m^{a_m}$ for primes $p_1, \ldots, p_m$ and integers $a_1, \ldots, a_m \geq 1$ then the longest chain has size $a_1 + a_2 + \cdots + a_m + 1$.

The exponent b_i of p_i in a chain beginning at 1 satisfies $0 \leq b_i \leq a_i$: it begins 0 and then forms a non-decreasing sequence, until it finally reaches a_i (its maximum value among divisors of n). Consider now the exponents $b_1, b_2, \ldots, b_m$ of the primes $p_1, p_2, \ldots, p_m$ in a divisor of n together while moving up a chain in $(D_n, \preceq)$. From one to the next divisor at least one of the exponents b_i must increase by 1 (or more). The number of such increments to the ith exponent is bounded above by a_i. Hence the total number of steps is bounded above by $\sum a_i$, making the size of the chain at most $1 + \sum a_i$. By incrementing just one exponent b_i by 1 each time, this bound can be achieved.

What is the size of the largest antichain in $(D_n, \preceq)$?

This last question should not have remained on the exercise sheet! Answering it is difficult, and includes Sperner's theorem as a special case. For information, here is the answer (for a proof see N. G. de Bruijn, Ca. van Ebbenhorst Tengbergen, and D. Kruyswijk, On the set of divisors of a number, Nieuw Arch. Wiskunde (2) 23 (1951), 191–193):

Let $n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_m^{a_m}$ for primes $p_1, \ldots, p_m$ and integers $a_1, \ldots, a_m \geq 1$ and set $a = a_1 + a_2 + \cdots + a_m = \sum a_i$.

If $a = 2b$ is even, an antichain of divisors of n of maximum size includes the set of all divisors $p_1^{b_1} p_2^{b_2} \cdots p_m^{b_m}$ with $\sum b_i = b$. There may be other sets of divisors than these that also form an antichain of maximum size: for example $n = 24 = 2^3 \cdot 3$ has 6 antichains of maximum size 2:

$$\{4, 6\} = \{2^2, 2 \cdot 3\}, \quad \{8, 6\}, \{2, 3\}, \{4, 3\}, \{8, 3\}, \{8, 12\}.$$

(However, if $a_i = 1$ for each i then there is a unique antichain of maximum size.)

If $a = 2b + 1$ is odd, two antichains of divisors of n of maximum size include the set of divisors $p_1^{b_1} p_2^{b_2} \cdots p_m^{b_m}$ with $\sum b_i = b$, and the set of divisors $p_1^{b_1} p_2^{b_2} \cdots p_m^{b_m}$ with $\sum b_i = b+1$. (If $a_i = 1$ for each i then these two antichains are the only ones of maximum size.)

Recall from part (d) that the poset of divisors of $n = p_1 p_2 \cdots p_m$, where $p_1, p_2, \ldots, p_m$ are distinct primes, is isomorphic to the poset $([m], \subseteq)$.

Sperner's theorem The largest antichain in $([m], \subseteq)$ when m is even is $\binom{[m]}{m/2}$ and there are two largest antichains in $([m], \subseteq)$ when m is odd, namely $\binom{[m]}{(m-1)/2}$ and $\binom{[m]}{(m+1)/2}$.

[*Hint: give your answer in terms of the factorization of n into a product of prime powers. A* prime power *is a number of the form p^a for some prime p and integer $a \geq 1$. For a number $n > 1$ we have $n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_m^{a_m}$ for primes $p_1, \ldots, p_m$ and integers $a_1, \ldots, a_m \geq 1$. For the above examples, $8 = 2^3$ and $14 = 2 \cdot 7$.*]