Discrete Mathematics

Exercise sheet 3

17 /20 October 2016

Notation: $[n] = \{1, 2, \ldots, n\}$.

1.

(a) State how many functions there are from $[n]$ to $[m]$, where $m, n \in \mathbb{N}$.

(b) Deduce from your answer to (a) that there are 2^n subsets of $[n]$.

(c) Determine the number of ordered pairs (A, B), where $A \subseteq B \subseteq [n]$.

(d) Determine the number of ordered triples (A, B, C), where $A \subseteq B \subseteq C \subseteq [n]$.

2. A permutation of $[n]$ is a bijection $f : [n] \to [n]$.

(a) Look up/remind yourself what is meant by a *cycle* of the permutation f (e.g. section 3.2 of Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, page 65 in 2nd ed).

(b) How many permutations of $[n]$ have a single cycle?

(c) For a permutation $f : [n] \to [n]$, define the k-fold composition of f recursively by $f^1 = f$ and $f^k = f^{k-1} \circ f$. Let R be the relation on $[n]$ defined by $(x, y) \in R$ if and only if there exists an integer $k \geq 1$ such that $f^k(x) = y$.

Prove that the relation R is reflexive, symmetric and transitive.

3. Let $\binom{n}{k}$ denote the number of subsets of k elements from $[n]$. (For $n \geq 0$ we have $\binom{n}{0} = 1 = \binom{n}{n}$.)

Prove the following identities by using this combinatorial definition of $\binom{n}{k}$:

(a) $\binom{n}{n-k} = \binom{n}{k}$ for $0 \leq k \leq n$.

(b) $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ for $1 \leq k \leq n-1$.

(c)

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

(d)

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$