Discrete Mathematics

Exercise sheet 11

13/20 December 2016

1. A *permutation matrix* is a matrix in which each entry is either 0 or 1 and each row and column contains precisely one 1.

(a) Let $P = (p_{ij})$ be an $n \times n$ permutation matrix whose rows and columns are indexed by $[n]$. Define the function $f : [n] \to [n]$ by $f(i) = j$ precisely when $p_{ij} = 1$.

Explain briefly why f is a permutation of $[n]$.

For each $i \in [n]$ there is unique j such that $p_{ij} = 1$. This is to say that $f : i \mapsto j$ (for row i select the column j in which the entry 1 appears) is an injection $[n] \to [n]$, and therefore a bijection. A permutation of $[n]$ is by definition a bijection $[n] \to [n]$.

[Observe that $f^{-1} : j \mapsto i$ (for column j select the row i in which the entry 1 appears) defines the inverse permutation.]

(b) Prove that

$$P \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_{f(1)} \\ x_{f(2)} \\ \vdots \\ x_{f(n)} \end{pmatrix},$$

and deduce that

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} P^T = \begin{pmatrix} x_{f(1)} & x_{f(2)} & \cdots & x_{f(n)} \end{pmatrix}.$$

Let

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

The ith entry of the column vector $P\mathbf{x}$ is given by

$$(P\mathbf{x})_i = \sum_j p_{ij} x_j = x_{f(i)}.$$

as $p_{i,j} = 0$ if $j \neq f(i)$ and $p_{i,f(i)} = 1$. This yields the equation

$$P \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_{f(1)} \\ x_{f(2)} \\ \vdots \\ x_{f(n)} \end{pmatrix},$$

the transpose of which is

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} P^T = \begin{pmatrix} x_{f(1)} & x_{f(2)} & \cdots & x_{f(n)} \end{pmatrix},$$

(in which we use the fact that $(AB)^T = B^T A^T$ for any matrices A and B).

(c) Prove that G and G' are isomorphic graphs if and only if a permutation matrix P exists such that

$$A_{G'} = PA_GP^T.$$

where A_G is the adjacency matrix of G and $A_{G'}$ is the adjacency matrix of G'.

Let

$$A_G = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots & \cdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

and P be defined as in (a).

By (b),

$$P\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots & \cdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} a_{f(1)1} & a_{f(1)2} & \cdots & a_{f(1)n} \\ a_{f(2)1} & a_{f(2)2} & \cdots & a_{f(2)n} \\ & \cdots & \cdots & \\ a_{f(n)1} & a_{f(n)2} & \cdots & a_{f(n)n} \end{pmatrix},$$

and by the transpose version of (b),

$$\begin{pmatrix} a_{f(1)1} & a_{f(1)2} & \cdots & a_{f(1)n} \\ a_{f(2)1} & a_{f(2)2} & \cdots & a_{f(2)n} \\ & \cdots & \cdots & \\ a_{f(n)1} & a_{f(n)2} & \cdots & a_{f(n)n} \end{pmatrix} P^T = \begin{pmatrix} a_{f(1)f(1)} & a_{f(1)f(2)} & \cdots & a_{f(1)f(n)} \\ a_{f(2)f(1)} & a_{f(2)f(2)} & \cdots & a_{f(2)f(n)} \\ & \cdots & \cdots & \\ a_{f(n)f(1)} & a_{f(n)f(2)} & \cdots & a_{f(n)f(n)} \end{pmatrix}.$$

There is an isomorphism $G \cong G'$ if and only if, identifying the vertex set of G and vertex set of G' with $[n]$, there is a bijection $f : [n] \to [n]$ such that $ij \in E(G)$ iff $f(i)f(j) \in E(G')$. In terms of the adjacency matrix A_G, $G \cong G'$ if and only if there is a permutation f of $[n]$ such that $A_{G'} = (a_{f(i)f(j)})$. In other words, $A_{G'} = PA_GP^T$ for some permutation matrix P as in the above equation.

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, section 4.2, exercise 4.2.12.]

2. A *tree* is a connected graph containing no cycles as a subgraph.

(a) Prove that if $G = (V, E)$ is a graph containing no cycles and satisfying $|V| = |E| + 1$ then G is a tree.

We just need to prove that G is connected. We proceed by induction on $|V|$. When $|V| = 1$ we have $G \cong K_1$, which is connected.

Assume as inductive hypothesis that any cycle-free graph on $|V| = n$ vertices and $n - 1$ edges is a tree. Suppose G is a cycle-free graph on $n + 1$ vertices and $n \geq 1$ edges.

Since there are no cycles and at least one edge there must be a vertex v of degree 1 in G, adjacent to vertex u. (Consider a connected component of G, which is a tree, since it has no cycles and is connected, and we know a tree with at least one edge has two or more vertices of degree 1.)[1]

Then by induction hypothesis $G - v$ is a tree (it has n vertices and $n - 1$ edges and contains no cycles). The graph G obtained by adding v and its single incident edge uv to $G - v$ retains the property of having no cycles and is connected (to get from v to any vertex there is a path through u).

Hence G is a tree and the induction step goes through.

[Alternative proof using the result of 2(c): consider the connected components of G containing no cycles: each of these is a tree (U, F), for which we know $|U| = |F| + 1$. If there are $k \geq 1$ connected components then we have $|V| = |E| + k$, so $k = 1$ and G is connected.]

[1] Or, repeating the standard proof given for trees, consider a maximal path beginning at a vertex of degree at least 1 - this must end in a vertex of degree 1, for otherwise the path could either be continued (contradicting maximality) or there would be an edge from the endpoint of the path to one of its other vertices forming a cycle (contradicting the assumption that there are no cycles).

(b) Prove that if $G = (V, E)$ is a connected graph and $|V| = |E| + 1$ then G is a tree.

Consider a connected graph G satisfying $|V| = |E| + 1 \geq 2$. The sum of the degrees of all vertices is thus $2|E| = 2|V| - 2$. This means (as in (a)) that not all vertices can have degree 2 or larger, and since all the degrees are at least 1 (by connectedness), there exists a vertex v of degree exactly 1. The graph $G - v$ is again connected and it satisfies $|V(G - v)| = |E(G - v)| + 1$. Hence it is a tree by the inductive hypothesis, and thus G is a tree as well.

(c) Prove the converse of (b).

The converse statement to (b) is that if $G = (V, E)$ is a tree (connected and no cycles) then G is connected and $|V| = |E| + 1$.

We wish to prove that if $G = (V, E)$ is connected with no cycles then $|V| = |E| + 1$.

The statement is true for $|V| = 1$, when $G = K_1$.

We may assume G is a tree with at least two vertices. Let v be a vertex of degree 1 in G, adjacent to vertex u. Suppose as inductive hypothesis that the tree $G - v$ satisfies the conclusion that $|V(G-v)| = |E(G-v)| - 1$. Then $|V(G)| = |V(G-v)| + 1$ and $|E(G)| = |E(G-v)| + 1$ (adding a vertex of degree 1 increases number of vertices and edges both by 1), whence $|V(G)| = |E(G)| - 1$, establishing the induction step.

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, section 5.1, Theorem 5.1.2(v) and exercise 5.1.2]