Combinatorics and Graph Theory I

Exercise sheet 8: finite projective planes

26 April 2017

1. Prove that the Fano plane is the only projective plane of order 2 (i.e. any projective plane of order 2 is isomorphic to it—define an isomorphism of set systems first).

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, 2nd ed., 9.1, exercise 1]

2. Let $(X, \mathcal{L})$ be a finite projective plane with set of points X and set of lines $\mathcal{L}$. Let r be the *order* of $(X, \mathcal{L})$, defined as the number of points less one in any given line, i.e., $r = |L| - 1$ for $L \in \mathcal{L}$. The *incidence graph* (or *Levi graph*) of $(X, \mathcal{L})$ is the bipartite graph on $X \cup \mathcal{L}$ with an edge joining x to L precisely when $x \in L$. (See Figure 1 for the Levi graph of the Fano plane, drawn more symmetrically than in Matoušek & Nešetřil, 2nd ed., Figure 9.3.)

(i) The *girth* of a graph with at least one cycle is the smallest positive integer g for which there is a g-cycle. (Thus for instance a triangle-free graph has girth at least 4.) A *k-regular graph* is a graph in which each vertex has degree k.

Show that a k-regular graph with girth $2m+1$ must have at least $1 + k + k(k-1) + \cdots + k(k-1)^{m-1}$ vertices, and that a k-regular graph with girth $2m$ must have at least $2[1 + (k-1) + (k-1)^2 + \cdots + (k-1)^{m-1}]$ vertices.

(ii) Show that the incidence graph of $(X, \mathcal{L})$ is an $(r+1)$-regular graph of girth 6 which attains the lower bound given in (i) for $m = 3$. *(Thus the incidence graph of a projective plane of order r has the minimum number of vertices among all $(r+1)$-regular graphs of girth 6.)*

[N. Biggs, *Discrete Mathematics,* rev. ed., 1989. 8.8, exercises 18 (and 20), and 16.10, exercise 10.]

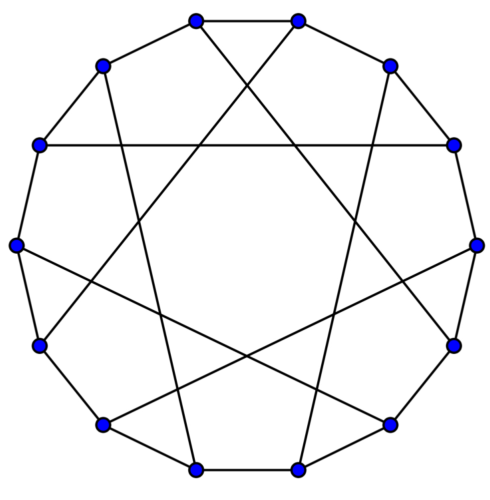

Figure 1: The Heawood graph, incidence graph of the Fano plane. (Image source: Wikipedia.)

3. Let X be a finite set and $\mathcal{L}$ a system of lines (subsets of X) satisfying conditions (P1) and (P2), and the following condition :

(P0') There exist at least two distinct lines having at least three points each.

Prove that any such $(X, \mathcal{L})$ is a finite projective plane. [*Hint*: By (P0') and (P1) the symmetric difference of the two such lines contains at least four points. Show these give a set F satisfying (P0).]

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, 2nd ed. 9.1, exercise 4]

4.

(i) Find an example of a set system $(X, \mathcal{L})$ on a non-empty finite set X that satisfies conditions (P1) and (P2) but does not satisfy (P0).

(ii) Describe *all* set systems $(X, \mathcal{L})$ on non-empty finite set X satisfying conditions (P1) and (P2) but not (P0).

[*Hint*: By question 3, it may be assumed that there is a most one line containing three or more points.]

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, 2nd ed. 9.1, exercise 3]

5. A *quadrangle* in a projective plane is a set of four points, no three of which are collinear. (The axiom (P0) says there is at least one quadrangle.)

(i) Show that there are exactly seven quadrangles in the Fano plane (projective plane of order 2). What is the the relationship between the seven quadrangles and the seven lines?

(ii) Suppose a, b, c, d are the points of a quadrangle and that $x = \overline{ab} \cap \overline{cd}$, $y = \overline{ac} \cap \overline{bd}$, and $z = \overline{ad} \cap \overline{bc}$. (The points x, y, z are known as the *diagonal points* of the quadrangle.)

Show that in the Fano plane the diagonal points of any quadrangle are collinear.

(iii) Give an example to show (ii) does not hold in the projective plane of order 3, i.e., produce a quadrangle in this projective plane of 13 points and lines whose diagonal points are not collinear.

[N. Biggs, *Discrete Mathematics,* rev. ed., 1989. Section 16.7, exercises 2,3,4.]