Combinatorics and Graph Theory I

Exercise sheet 7: Spanning trees and double counting

19 April 2017

1. Let $\tau(G)$ denote the number of spanning trees of a connected graph G. Cayley's formula states that $\tau(K_n) = n^{n-2}$ for $n \geq 2$. Let K_n^- denote a graph isomorphic to K_n with one edge removed.

Find a formula for $\tau(K_n^-)$.

[*Hint*: the number of spanning trees containing a given edge of K_n is by symmetry the same for all edges.]

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, 2nd ed., 8.1, exercise 2]

2.

(i) Determine natural numbers a and b with $a+b=n$ for which the product ab is maximized.

(ii) For natural numbers k and n, determine all values of natural numbers $a_1, \ldots, a_k$ satisfying $\sum_{i=1}^k a_i = n$ such that the product $a_1 a_2 \cdots a_k$ is maximized.

(iii) A complete k-partite graph $K(V_1, V_2, \ldots, V_k)$ on a vertex set V is determined by a partition $V_1, \ldots, V_k$ of the set V, in which edges are pairs $\{x, y\}$ of vertices such that x and y lie in different classes of the partition. Formally, $K(V_1, \ldots, V_k) = (V, E)$, where $\{x, y\} \in E$ exactly if $x \neq y$ and $|\{x, y\} \cap V_i| \leq 1$ for all $i = 1, \ldots, k$. Using part (ii), prove that the maximum number of edges of a complete k-partite graph on a given vertex set corresponds to a partition with almost equal parts, i.e. one with $||V_i| - |V_j|| \leq 1$ for all i, j. How many edges are there in such a graph $K(V_1, \ldots, V_k)$?

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, 2nd ed., 4.7, exercises 1,2 and 3]

3. Prove that for any $t \geq 2$, the maximum number of edges of a graph on n vertices containing no $K_{2,t}$ as a subgraph is at most

$$\frac{1}{2}\left(\sqrt{t-1}\,n^{3/2} + n\right).$$

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, 2nd ed. 7.3, exercise 1]

4. For real numbers $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ the Cauchy–Schwarz inequality states that

$$\sum_{i=1}^n x_i y_i \leq \sqrt{\sum_{i=1}^n x_i^2}\sqrt{\sum_{i=1}^n y_i^2}.$$

(i) Prove the Cauchy–Schwarz inequality by induction on n (square both sides first).

(ii) Prove the Cauchy–Schwarz inequality directly, starting from the inequality

$$\sum_{i,j=1}^{n} (x_i y_j - x_j y_i)^2 \geq 0.$$

(iii)

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, 2nd ed., 7.3 exercise 4]