Combinatorics and Graph Theory I

Exercise sheet 1: Estimates

22 February 2017

1. Show that if $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$ then $f_1(n) + f_2(n) = O(g_1(n) + g_2(n))$ and $f_1(n)f_2(n) = O(g_1(n)g_2(n))$.

Express in words the statements $f(n) = O(1)$, $g(n) = \Omega(1)$ and $h(n) = n^{O(1)}$.

(a) Prove that $n^\alpha = O(n^\beta)$ for $\alpha \leq \beta$.

(b) Prove that $n^\gamma = O(a^n)$ for any $a > 1$.

(c) Deduce from (b) that $(\ln n)^\gamma = O(n^\alpha)$ for any $\alpha > 0$.

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, section 3.4, Fact 3.4.3 and exercise 3.4.6.]

2. Prove using the Mean Value Theorem that $1 + x \leq e^x$ for all $x \in \mathbb{R}$.

[*Use the fact that the function* $f(x) = e^x$ *is its own derivative,* $f'(x) = e^x$*, and consider this function on the interval* $[0, x]$.]

(a) Prove by induction Bernouilli's Inequality $(1 + x)^n \geq 1 + nx$ for all $x \geq -1$,

(b) $e\left(\frac{n}{e}\right)^n \leq n!$ by induction on n,

(c) $n! \leq en\left(\frac{n}{e}\right)^n$ by induction on n,

(d) $n! \leq e\left(\frac{n+1}{e}\right)^{n+1}$ by taking natural logarithms and comparing $\ln n!$ with the integral $\int_1^{n+1} \ln x \, \mathrm{d}x$, and after this derive (b) from this inequality.

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, section 3.5, exercises 3.5.11 and 3.5.9, first and second proofs of Theorem 3.5.5]

3.

(a) Prove using integration that for $n \geq 1$,

$$\ln(n + 1) < \sum_{k=1}^{n} \frac{1}{k} \leq \ln n + 1.$$

[*Use the fact that if* $\int f(x)\,\mathrm{d}x = F(x) + c$ *for constant* c *then* $\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$. *Also that the area under the curve* $y = f(x)$ *between the lines* $x = a$ *and* $x = b$ *equals the integral* $\int_a^b f(x)\,\mathrm{d}x$.]

(b) Derive a similar estimate as (a) for the series $\sum_{k=1}^{n} \frac{1}{k^p}$ for $p > 1$.

(c) By considering the series $\sum a_k$ with terms

$$a_k = \frac{1}{k} - \int_k^{k+1} \frac{\mathrm{d}x}{x}$$

show that

$$\sum_{k=1}^{n} \frac{1}{k} = \ln n + \gamma + O(\frac{1}{n}),$$

where γ is the Euler-Mascheroni constant, $0 < \gamma < \sum_{k=1}^{\infty} \frac{1}{2k^2}$. [*Use the Taylor expansion for* $\ln(1+x)$ *with* $x = \frac{1}{k}$ *to bound* a_k*, express* $\sum \frac{1}{k}$ *in terms of* $\sum a_k$ *and an integral.*]

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, section 3.6, exercise 3.6.13(b) extended]

4.

(a) Prove the binomial expansion

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

(b) Use the binomial expansion to show that $\binom{n}{k} \le \left(\frac{en}{k}\right)^k$.

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, section 3.6, Theorem 3.6.1]

5.

(a) Prove the arithmetic-geometric mean inequality $\sqrt{ab} \le \frac{1}{2}(a+b)$.

(b) Prove by induction on n and using (a) that for $n \ge 1$ we have

$$2\sqrt{n+1} - 2 < \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \le 2\sqrt{n} - 1.$$

[*This can alternatively be obtained by integration as in question 3(b).*]

(c) Use the inequality $1 + x \le e^x$ and induction to prove the inequality in 3(a)

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, section 3.5, exercises 3.5.12 and 3.5.13.]