Combinatorics and Graph Theory I

Exercise sheet 4: Bipartite matching, Connectivity

19 March 2018 (to hand in 26 March)

References: Chapter III (Flows, Connectivity and Matching), Section 2 (Connectivity and Menger's Theorem) and Section 3 (Matching) in Bollobás *Modern Graph Theory*, 1998. Also, Chapter 10 (Digraphs, Networks and Flows) in Biggs, *Discrete Mathematics*, 1st ed. 1985 (2nd ed. 2002). and Chapters 2 and 3 (Matching, Connectivity) in Diestel, *Graph Theory* 2nd ed., 1997 (online ed. 2000).

1. An *augmenting path* with respect to a matching M in a bipartite graph G is a path P in G which starts at an unmatched vertex and then contains alternately an edge not in M and an edge in M and ends in another unmatched vertex. The symmetric difference of the edges of P and the edges of M is again a matching and covers two more vertices than M (the endpoints of P).

(i) Let M be a matching in a bipartite graph G. Show that if M contains fewer edges than some other matching N in G then G contains an augmenting path with respect to M.

[Consider the symmetric difference of M and N.]

(ii) Describe an algorithm based on (i) that finds a matching of maximum cardinality in any given biparite graph. [*Fine details not required; you may assume there is an oracle that provides you with an augmenting path when one exists.*]

[Diestel, *Graph Theory* 2nd ed., 2.1, chapter 2 exercises 1, 2]

2.

(a) Use Hall's condition to show that the bipartite graph in Figure 1 has no complete matching.

(b) Let M be the matching $\{x_3y_2, x_4y_4, x_5y_5\}$ denoted by heavier lines in Figure 1.

(i) Find an alternating path for M beginning at x_2.

(ii) Use it to construct a matching M' with four edges.

(iii) Check that there is no alternating path for M'.

(iv) Is M' a maximum matching? (i.e. are there any matchings with more than four edges?)

[Biggs, *Discrete Mathematics*, exercises 10.4.1 and 10.4.2]

3. Let G be a bipartite graph with vertex sets V_1 and V_2. Let U be the set of vertices of maximal degree (i.e., the degree of each vertex in U is the maximum degree of G). Show that there is a complete matching from $U \cap V_1$ into V_2.

[Bollobás, *Modern Graph Theory*, III.6 exercise 21.]

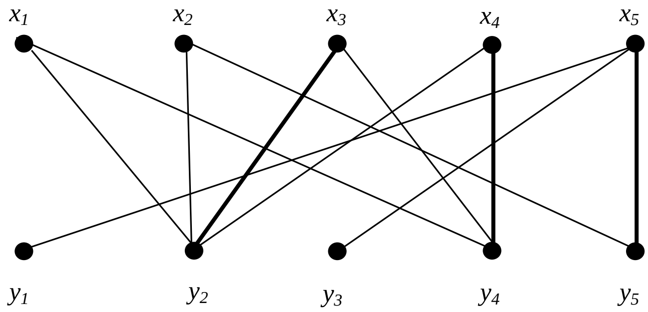

Figure 1: Bipartite graph for Exercise 2.

4. Let $\delta(G)$ denote the minimum degree of graph G.

(i) Define the parameters $\kappa(G)$ and $\lambda(G)$.

(ii) Prove that

$$\kappa(G) \leq \lambda(G) \leq \delta(G)$$

for a graph G on more than one vertex.

[Bollobás, *Modern Graph Theory*, III.2.]

(iii) Let k and ℓ be arbitrary integers with $1 \leq k \leq \ell$.

(a) Construct for any given such k, ℓ a graph G with $\kappa(G) = k$ and $\lambda(G) = \ell$.

(b) Construct for any given such k, ℓ a graph G with $\kappa(G) = k$ and $\kappa(G - v) = \ell$ for some vertex v.

[Bollobás, *Modern Graph Theory*, III.6, exercise 11]

5. Given $U \subset V(G)$ and a vertex $x \in V(G) - U$, an $x - U$ *fan* is a set of $|U|$ paths from x to U any two of which have exactly the vertex x in common. Prove that a graph G is k-connected iff $|G| \geq k + 1$ and for any $U \subset V(G)$ of size $|U| = k$ and vertex x not in U there is an $x - U$ fan in G.

[*Given a pair (x, U), add a vertex u to G and join it to each vertex in U. Check that the new graph is k-connected if G is. Apply Menger's theorem for x and u.*]

[Bollobás, *Modern Graph Theory*, III.6 exercise 13]