# Non-embeddability of buildings


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joint work with

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# Simplicial complexes

## Definition

An (abstract) simplicial complex $K$ with vertex set $V$ is a hereditary set system ($K \subseteq 2^V$), that is, if $A \in K$ and $B \subseteq A$, then $B \in K$). Elements of $K$ are faces of $K$.

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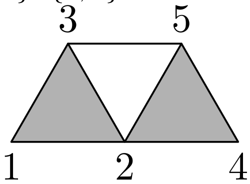
### Geometric realization of a simplicial complex

Maximal faces of $K$: $\{1, 2, 3\}$, $\{2, 4, 5\}$, $\{3, 5\}$.

A geometric realization $|K|$:



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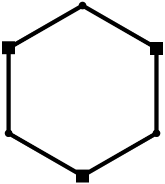
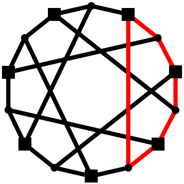
- Composed of smaller subcomplexes (apartments) that are Coxeter complexes.
- A finite Coxeter complex is a symmetric triangulation of a sphere (the symmetry corresponds to some Coxeter group).
- In order to get a building, apartments are glued together in such a way that there many automorphisms of the building sending an apartment to another apartment.
- For our purposes, we do not define buildings precisely. We use a well known classification of buildings (described later on). They fall into several types: infinite families of type A, B, and D and also few other sporadic cases.

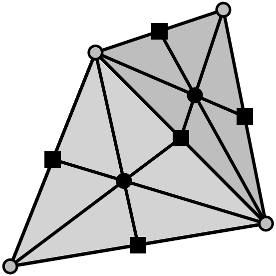
# Examples



A 1-dim. Coxeter complex



A 1-dim. building (Fano plane)



A 2-dimensional Coxeter complex

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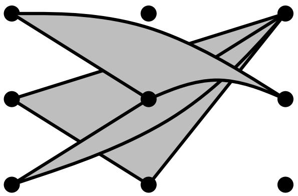
- Embedding 1-dimensional complexes into $\mathbb{R}^2 \equiv$ graph planarity.
- Every $k$-dimensional simplicial complex embeds into $\mathbb{R}^{2k+1}$ by a general position argument.
- There are $k$-dimensional simplicial complexes that do not embed into $\mathbb{R}^{2k}$: e.g., the $k$-skeleton of $(2k+2)$-simplex or the $(k+1)$-tuple join of the three-point discrete set $D_3^{*(k+1)}$.

$D_3^{*3}$



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**Theorem**

*A $d$-dimensional thick building $\Delta$ does not embed into $\mathbb{R}^{2d}$ if*

1. $d = 1$,
2. *$\Delta$ is a type $A$ building,*
3. *$\Delta$ is a type $B$ building coming from an alternating bilinear form on $\mathbb{F}_q^{2d+2}$, or*
4. *$\Delta$ is a type $B$ building coming from an Hermitian form on $\mathbb{F}_{q^2}^{2d+2}$ or $\mathbb{F}_{q^2}^{2d+3}$.*

# 1-dimensional buildings

## Proposition (classical)

*A finite connected graph $G$ is a building if and only if $G$ is bipartite, $\delta(G) \geq 2$, $\operatorname{diam}(G) = m$ and $g(G) = 2m$ for some $m \geq 2$ where $\delta(G)$ denotes the minimum degree and $g(G)$ denotes the girth.*

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Thick 1-dimensional buildings are non-planar:

- If $m \geq 3$, then there are no planar graphs with $\delta(G) \geq 3$ and $g(G) \geq 6$ by Euler's formula.
- If $m = 2$ then one can easily derive that $G = K_{k,\ell}$ for $k, \ell \geq 3$.

# Type A buildings

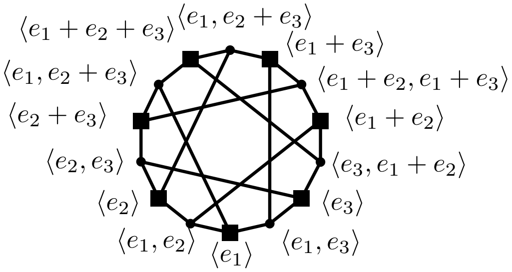
### Definition

Given a poset $P$, the order complex of $P$, denoted by $\Delta(P)$ is a simplicial complex with the vertex set $P$ such that the faces of $\Delta(P)$ are chains in $P$.

### Definition

By projective geometry we mean the poset $P_q^d$ of all proper subspaces $X$ of $\mathbb{F}_q^d$, where by proper we mean $\dim X \notin \{0, d\}$. A $d$-dimensional building of type $A$ is the order complex $\Delta(P_q^{d+2})$.

$\Delta(P_2^{1+2})$ :



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Given a simplicial complex $K$ of dimension $k$, there is a certain cohomology obstruction $\vartheta(K)$ called Van Kampen's obstruction such that

- if $\vartheta(K) \neq 0$ then $K$ does not embed into $\mathbb{R}^{2k}$, and also
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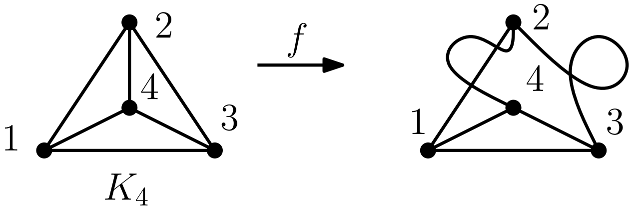
## Remark

- If $k$ is fixed, then $\vartheta(K)$ is computable in a time polynomial in the size of $K$.
- We have $\vartheta(D_3^{*(k+1)}) \neq 0$.

# Almost injective maps

## Definition

Let $K$ be a simplicial complex, $X$ be a topological space and $f\colon |K| \to X$ be a map. We say that $f$ is almost injective if and only if for every two **disjoint** faces $\sigma, \tau \in K$ their images $f(|\sigma|)$ and $f(|\tau|)$ are disjoint as well.



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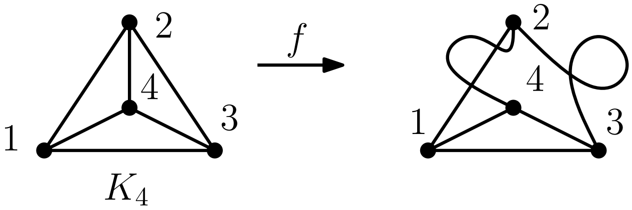


## Proposition (Van Kampen, Shapiro, Wu)

*Let $K$ be a $d$-dimensional simplicial complex with $\vartheta(K) \neq 0$. Let $L$ be another complex and $f\colon |K| \to |L|$ an almost injective map. Then $L$ does not embed into $\mathbb{R}^{2d}$.*

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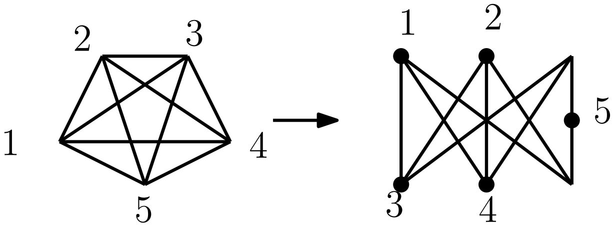
### Remark

Buildings of type B arise from the posets other than $P_q^{d+2}$; however, the approach is similar.

# Intermezzo

## Observation

*There is an almost injective map $f: K_5 \to K_{3,3}$:*



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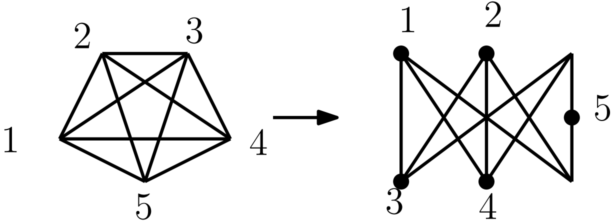


Therefore by Kuratowski's theorem and the proposition we have:

## Corollary

*A graph $G$ is non-planar if and only if there is an almost injective map $f\colon |K_5| \to |G|$. In addition, if $G$ is nonplanar, then $f$ can be taken "reasonably nice".*

# Questions and further research

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A possible approach:

### Question

Does the high symmetry of buildings imply that they are higher dimensional analogues of expander graphs?

### Question

Is there a notion of higher dimensional expansion of $d$-dimensional simplicial complexes such that all the complexes with high expansion do not embed into $\mathbb{R}^{2d}$.