Vector chromatic number – where graph theory meets semidefinite programming

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Outline

1. Intro – SDP, def’s
2. Why
3. My point of view
Semidefinite programming (SDP)

\[
X \succeq 0 \iff X \text{ is positive semidefinite (PSD)} \\
\iff \text{there are vectors } v_1, \ldots, v_n \text{ so that } X_{i,j} = v_i^T v_j \text{ (} X \text{ is Gram matrix)}
\]

- SDP: \( \inf C \cdot X : (\forall i) A_i \cdot X = b_i, X \succeq 0, X \in \mathbb{R}^{n \times n} \)
  - when all matrices are diagonal, we get classical linear programming
  - effectively solvable – for every \( \varepsilon > 0 \) one can in time polynomial to the size of the input and \( \log \frac{1}{\varepsilon} \) approximate solution with precision \( \varepsilon \).
  - in most cases, duality holds, similarly as for LP
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Vector coloring

– usual coloring of $C_5$

We are trying to assign unit vectors to vertices of the graph, so that adjacent vertices are far apart.
– (variant: strict vector coloring – all edges have to be of the same length)
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Vector coloring – definition

Definition (Karger, Motwani and Sudan, 1998)

**Given:** graph $G$ with $n$ vertices

**Find:** minimal $t < 0$ s.t. $\exists f : V(G) \rightarrow \mathbb{R}^n$

- $|f(v)| = 1 \quad \forall v \in V(G)$ and
- $\langle f(u), f(v) \rangle \leq t \quad \forall uv \in E(G)$.

$t(G) := \text{minimal such } t$.

$\chi_v(G) := 1 - \frac{1}{t(G)} \ldots \text{vector chromatic number}$

$t(G)$ is defined by a semidefinite program $\Rightarrow$ the minimum exists and can be approximated with an absolute error $< \varepsilon$ in time polynomial in $n$ and $\log \frac{1}{\varepsilon}$. 

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$\chi_{sv}(G) = \vartheta(\bar{G}) =: \bar{\vartheta}(G)$
Intro – SDP, def’s

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Original motivation – \( \Theta \)

**Definition (Shannon, 1940’s)**

Shannon capacity – communication capacity of a channel

\[
\Theta(G) = \lim_{n \to \infty} \sqrt[n]{\alpha(G \boxtimes G \boxtimes \cdots \boxtimes G)}
\]

Problem: \( \Theta(C_5) = ? \).

**Theorem (Lovász, 1979)**

*There is a graph parameter \( \vartheta(G) \), such that*

\[\begin{align*}
\vartheta(G) &\geq \alpha(G) \\
\vartheta(G \boxtimes H) &= \vartheta(G) \vartheta(H) \quad \Rightarrow \quad \vartheta(G) \geq \Theta(G) \\
\vartheta(C_5) &\text{ can be computed easily}
\end{align*}\]

\[\ldots \text{ in fact } \vartheta(G) \text{ can be computed for any } G \text{ in a polynomial time}\]
Original motivation – $\Theta$

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There is a graph parameter $\vartheta(G)$, such that

- $\vartheta(G) \geq \alpha(G)$
- $\vartheta(G \boxtimes H) = \vartheta(G) \vartheta(H)$ $\Rightarrow$ $\vartheta(G) \geq \Theta(G)$
- $\vartheta(C_5)$ can be computed easily

... in fact $\vartheta(G)$ can be computed for any $G$ in a polynomial time
Approximating chromatic number

**Theorem (Karger, Motwani, Sudan, 1998)**

\[ \chi_v(G) \leq k \implies \chi(G) \leq \Delta^{1-2/k+o(1)} \]

assuming \( k \) constant, \( \Delta = \Delta(G) \).

*Moreover*, such coloring can be found in a polynomial time.

**Theorem (Feige, Langberg, Schachtman, 2004)**

*This is basically best possible.*
Approximating chromatic number

**Theorem (Coja-Oghlan, 2005)**

*Existence of a k-coloring of G(n, p) can be tested in average polynomial time (provided pn ≥ ck^2).*

**Idea:** The probability that χ_v will not provide the desired bound is exponentially small. Thus in such case we can use some algorithm with exponential running time.

**Required ingredient:** understanding of the behaviour of χ_v(G(n, p)) and ϑ(G_{n,p}).
Approximating chromatic number

**Theorem (Coja-Oghlan, 2005)**

*Existence of a $k$-coloring of $G(n, p)$ can be tested in average polynomial time (provided $pn \geq ck^2$).*

**Idea:** The probability that $\chi_V$ will not provide the desired bound is exponentially small. Thus in such case we can use some algorithm with exponential running time.

**Required ingredient:** understanding of the behaviour of $\chi_V(G(n, p))$ and $\vartheta(G_{n,p})$. 
Approximating cubical chromatic number

Theorem (Š., 2010+)

For every graph $G$ we have

$$\chi_v(G) \leq \chi_q(G) \leq \frac{\pi}{2} \chi_v(G).$$

Here $\chi_q$ is the cubical chromatic number.
Vector chromatic number as a worthy graph parameter

Observation

\[ G \rightarrow H \Rightarrow \chi_v(G) \leq \chi_v(H) \]

(And the same is true for \(\chi_{sv} = \bar{\vartheta} \).)

Thus \(\chi_v\) is a “coloring-type” parameter and we may understand it as a mean to understand the homomorphism structure of graphs.

Compared to other such parameters (\(\chi, \chi_c, \chi_f, \ldots\)) this one is computationally tractable (there is a polynomial-time algorithm to approximate it).
Example 1: values for random graph

Coja-Oghlan – concentrated on an interval of width one
Computational evidence: converges to a normal distribution, if $p = \text{const.}$

$\chi_{sv}(G(100, .5))$

However, if $p = o(1)$, it seems that $\chi_v(G(n, p))$ converges to a two-point distribution, rather surprising phenomenon.
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$\chi_{sv}(G(200, .1))$
## Example 2: product (Hedetniemi) conjecture

### Observation

\[ G \times H \rightarrow G \]

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*Thus:* \( \chi(G \times H) \leq \min\{\chi(G), \chi(H)\} \).

### Conjecture (Hedetniemi, 1966)

\[ \chi(G \times H) = \min\{\chi(G), \chi(H)\} \]

Note: a simple exercise is to show that \( \chi(G \square H) = \max\{\chi(G), \chi(H)\} \): we have \( G, H \rightarrow G \square H \) and it is not hard to actually provide the coloring.
Example 2: product conjecture for $\bar{\vartheta}$

**Observation**

\[ G \times H \rightarrow G \]
\[ G \times H \rightarrow H \]

Thus: \( \bar{\vartheta}(G \times H) \leq \min\{\bar{\vartheta}(G), \bar{\vartheta}(H)\} \).

**Conjecture (Š., 2011)**

\( \bar{\vartheta}(G \times H) = \min\{\bar{\vartheta}(G), \bar{\vartheta}(H)\} \).
Example 2: product conjecture for $\overline{\vartheta}$ – approach

Compared to the normal chromatic number, we can use SDP duality – so that we have a certificate for $\chi_{\overline{\vartheta}}(G) \geq c$ (whereas for normal coloring one only has certificate for $\chi(G) \leq c$ – namely a coloring using $c$ colors).

Using this idea and some variants of $\overline{\vartheta}$, one can find that

$$\overline{\vartheta}(G \Box H) \overline{\vartheta}(G \times H) \geq \overline{\vartheta}(G) \overline{\vartheta}(H).$$

As $\overline{\vartheta}(G \Box H) \geq \max\{\overline{\vartheta}(G), \overline{\vartheta}(H)\}$, it’s enough to show also the reverse inequality for $\overline{\vartheta}(G \Box H)$. 
Example 2: product conjecture for \( \bar{\mathcal{V}} \) – matrix completion

\[ \ldots \text{it’s enough to show also the reverse inequality for } \bar{\mathcal{V}}(G \square H) \text{ (this was trivial for } \chi \text{)}. \]

It leads to the following conjecture about matrices:

**Conjecture (Š., 2012)**

*Let \( A, B \) be PSD matrices with 1’s on diagonals. Then

\[
I \otimes B + A \otimes I - I \otimes I
\]

*can be completed to a PSD matrix.*
Example 3 – graphs on surfaces?

- Can you show that $\chi_v(G) < 5$ for every planar graph $G$?
- Can you show version of the Hadwiger conjecture?