Vector chromatic number – where graph theory meets semidefinite programming

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Outline

Semidefinite programming (SDP)

$X \succcurlyeq 0 \Leftrightarrow X$ is positive semidefinite (PSD)
$\Leftrightarrow$ there are vectors $v_1, \dots, v_n$ so that $X_{i,j} = v_i^T v_j$ (X is Gram matrix)

- SDP: $\inf C \bullet X : (\forall i)\ A_i \bullet X = b_i, X \succcurlyeq 0, X \in \mathbb{R}^{n \times n}$
- when all matrices are diagonal, we get classical linear programming
- effectively solvable – for every $\varepsilon > 0$ one can in time polynomial to the size of the input and $\log \frac{1}{\varepsilon}$ approximate solution with precision ε.
- in most cases, duality holds, similarly as for LP

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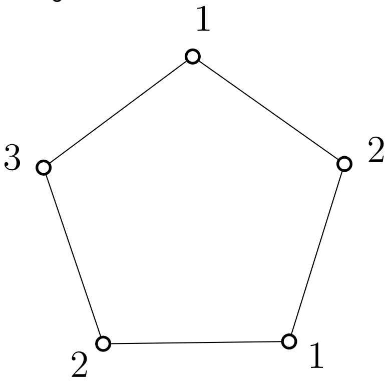
Vector coloring

– usual coloring of C_5



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We are trying to assign **unit vectors** to vertices of the graph, so that that adjacent vertices are **far apart.**

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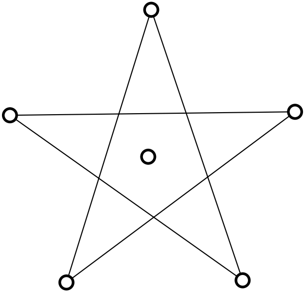

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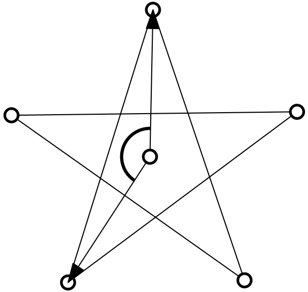

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Vector coloring – definition

Definition (Karger, Motwani and Sudan, 1998)

Given: graph G with n vertices
Find: minimal $t < 0$ s.t. $\exists f : V(G) \to \mathbb{R}^n$

- $|f(v)| = 1 \qquad \forall v \in V(G)$ and
- $\langle f(u), f(v) \rangle \leq t \qquad \forall uv \in E(G)$.

$t(G) :=$ minimal such t.
$\chi_v(G) := 1 - \frac{1}{t(G)}$ . . . *vector chromatic number*

$t(G)$ is defined by a semidefinite program $\Longrightarrow$ the minimum exists and can be approximated with an absolute error $< \varepsilon$ in time polynomial in n and $\log \frac{1}{\varepsilon}$.

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$$\chi_{sv}(G) = \vartheta(\bar{G}) =: \bar{\vartheta}(G)$$

Original motivation – Θ

Definition (Shannon, 1940's)

Shannon capacity – communication capacity of a channel

$$\Theta(G) = \lim_{n\to\infty} \sqrt[n]{\alpha(G \boxtimes G \boxtimes \cdots \boxtimes G)}$$

Problem: $\Theta(C_5) = ?$.

Theorem (Lovász, 1979)

There is a graph parameter $\vartheta(G)$, such that

- $\vartheta(G) \geq \alpha(G)$
- $\vartheta(G \boxtimes H) = \vartheta(G)\vartheta(H) \quad \Rightarrow \quad \vartheta(G) \geq \Theta(G)$
- $\vartheta(C_5)$ *can be computed easily*

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... in fact $\vartheta(G)$ can be computed for any G in a polynomial time

Approximating chromatic number

Theorem (Karger, Motwani, Sudan, 1998)

$$\chi_v(G) \le k \quad \Rightarrow \quad \chi(G) \le \Delta^{1-2/k+o(1)}$$

assuming k constant, $\Delta = \Delta(G)$.
***Moreover**, such coloring can be found in a polynomial time.*

Theorem (Feige, Langberg, Schachtman, 2004)

This is basically best possible.

Approximating chromatic number

Theorem (Coja-Oghlan, 2005)

Existence of a k-coloring of $G(n, p)$ can be tested in average polynomial time (provided $pn \geq ck^2$).

Idea: The probability that χ_V will not provide the desired bound is exponentially small. Thus in such case we can use some algorithm with exponential running time.

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Approximating cubical chromatic number

Theorem (Š., 2010+)

For every graph G we have

$$\chi_v(G) \le \chi_q(G) \le \frac{\pi}{2}\chi_v(G)\,.$$

Here χ_q is the cubical chromatic number.

Vector chromatic number as a worthy graph parameter

Observation

$$G \to H \quad \Rightarrow \chi_v(G) \le \chi_v(H)$$

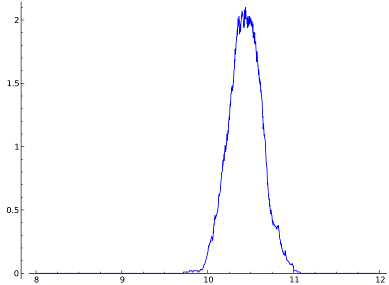
(And the same is true for $\chi_{sv} = \bar{\vartheta}$*.)*

Thus χ_v is a "coloring-type" parameter and we may understand it as a mean to understand the homomorphism structure of graphs.
Compared to other such parameters (χ, χ_c, χ_f, ...) this one is computationally tractable (there is a polynomial-time algorithm to approximate it).

Example 1: values for random graph

Coja-Oghlan – concentrated on an interval of width one
Computational evidence: converges to a normal distribution, if $p = const$.

$\chi_{sv}(G(100, .5))$



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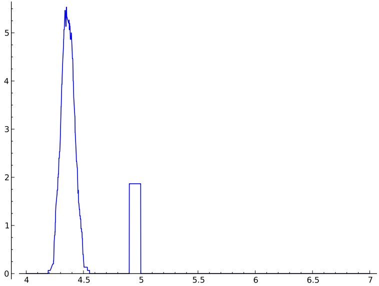

Example 2: product (Hedetniemi) conjecture

Observation

$$G \times H \to G$$

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Thus: $\chi(G \times H) \leq \min\{\chi(G), \chi(H)\}$.

Conjecture (Hedetniemi, 1966)

$\chi(G \times H) = \min\{\chi(G), \chi(H)\}$.

Note: a simple exercise is to show that $\chi(G \Box H) = \max\{\chi(G), \chi(H)\}$: we have $G, H \to G \Box H$ and it is not hard to actually provide the coloring.

Example 2: product conjecture for $\bar{\vartheta}$

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Thus: $\bar{\vartheta}(G \times H) \leq \min\{\bar{\vartheta}(G), \bar{\vartheta}(H)\}$.

Conjecture (Š., 2011)

$\bar{\vartheta}(G \times H) = \min\{\bar{\vartheta}(G), \bar{\vartheta}(H)\}$.

Example 2: product conjecture for $\bar{\vartheta}$ – approach

Compared to the normal chromatic number, we can use SDP duality – so that we have a certificate for $\chi_v(G) \geq c$ (whereas for normal coloring one only has certificate for $\chi(G) \leq c$ – namely a coloring using c colors).

Using this idea and some variants of $\bar{\vartheta}$, one can find that

$$\bar{\vartheta}(G \square H)\bar{\vartheta}(G \times H) \geq \bar{\vartheta}(G)\bar{\vartheta}(H)\,.$$

As $\bar{\vartheta}(G \square H) \geq \max\{\bar{\vartheta}(G), \bar{\vartheta}(H)\}$, it's enough to show also the reverse inequality for $\bar{\vartheta}(G \square H)$.

Example 2: product conjecture for $\bar{\vartheta}$ – matrix completion

...it's enough to show also the reverse inequality for $\bar{\vartheta}(G\Box H)$ (this was trivial for χ).
It leads to the following conjecture about matrices:

Conjecture (Š., 2012)

Let A, B be PSD matrices with 1's on diagonals. Then

$$I \otimes B + A \otimes I - I \otimes I$$

can be completed to a PSD matrix.

Example 3 – graphs on surfaces?

- Can you show that $\chi_v(G) < 5$ for every planar graph G?
- Can you show version of the Hadwiger conjecture?