Basis stability in interval linear programming

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Introduction

Linear programming

Three basic forms of linear programs

$$
\begin{aligned}
f(A,b,c) &\equiv \min c^T x \ \text{ subject to } \ Ax = b,\ x \geq 0,\\
f(A,b,c) &\equiv \min c^T x \ \text{ subject to } \ Ax \leq b,\\
f(A,b,c) &\equiv \min c^T x \ \text{ subject to } \ Ax \leq b,\ x \geq 0.
\end{aligned}
$$

Notation

An interval matrix

$$\mathbf{A} := [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m\times n} \mid \underline{A} \leq A \leq \overline{A}\}.$$

The center and radius matrices

$$A_c := \frac{1}{2}(\overline{A} + \underline{A}), \quad A_\Delta := \frac{1}{2}(\overline{A} - \underline{A}).$$

Introduction

Interval linear programming

Family of linear programs with $A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in \mathbf{c}$, in short

$$f(\mathbf{A}, \mathbf{b}, \mathbf{c}) \equiv \min \mathbf{c}^T x \ \text{ subject to } \ \mathbf{A}x \overset{(\leq)}{=} \mathbf{b}, \ (x \geq 0).$$

A *realization* is a concrete linear program in this family.

The three forms are not transformable between each other!

Goals

- determine the optimal value range;
- determine a tight enclosure to the optimal solution set.

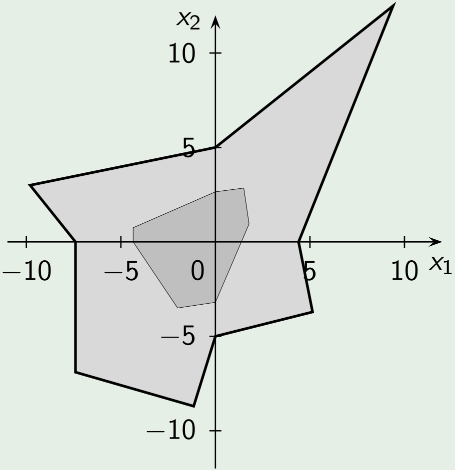
Introduction

Applications

- real-life problems affected by uncertainties
  - economics (portfolio selection,. . .)
  - environmental management (water resource and waste mng. planning)
  - logistic
  - . . .
- technical tool in constraint programming and global optimization
- others
  - interval matrix games
  - measure of sensitivity of linear programs

Example

Example (An interval polyhedron)



$$\begin{pmatrix} -[2,5] & -[7,11] \\ [1,13] & -[4,6] \\ [5,8] & [-2,1] \\ -[1,4] & [5,9] \\ -[5,6] & -[0,4] \end{pmatrix} x \leq \begin{pmatrix} [61,63] \\ [19,20] \\ [15,22] \\ [24,25] \\ [26,37] \end{pmatrix}$$

- union of all feasible sets in light gray,
- intersection of all feasible sets in dark gray,

Complexity of basic problems

	$\mathbf{A}x = \mathbf{b},\ x \geq 0$	$\mathbf{A}x \leq \mathbf{b}$	$\mathbf{A}x \leq \mathbf{b},\ x \geq 0$
strong feasibility	co-NP-hard	polynomial	polynomial
weak feasibility	polynomial	NP-hard	polynomial
strong unboundedness	co-NP-hard	polynomial	polynomial
weak unboundedness	suff. / necessary conditions only	suff. / necessary conditions only	polynomial
strong optimality	co-NP-hard	co-NP-hard	polynomial
weak optimality	suff. / necessary conditions only	suff. / necessary conditions only	suff. / necessary conditions only
optimal value range	$\underline{f}$ polynomial $\overline{f}$ NP-hard	$\underline{f}$ NP-hard $\overline{f}$ polynomial	polynomial

Optimal value range

Definition

$$\underline{f} := \min f(A, b, c) \ \text{ subject to } \ A \in \mathbf{A},\ b \in \mathbf{b},\ c \in \mathbf{c},$$
$$\overline{f} := \max f(A, b, c) \ \text{ subject to } \ A \in \mathbf{A},\ b \in \mathbf{b},\ c \in \mathbf{c}.$$

Theorem (Rohn, 2006)

We have for type $(\mathbf{A}x = \mathbf{b},\ x \geq 0)$

$$\underline{f} = \min \underline{c}^T x \ \textit{ subject to } \ \underline{A}x \leq \overline{b},\ \overline{A}x \geq \underline{b},\ x \geq 0,$$
$$\overline{f} = \max_{p \in \{\pm 1\}^m} f(A_c - \mathrm{diag}(p)\, A_\Delta, b_c + \mathrm{diag}(p)\, b_\Delta, \overline{c}).$$

Theorem (Vajda, 1961)

We have for type $(\mathbf{A}x \leq \mathbf{b},\ x \geq 0)$

$$\underline{f} = \min \underline{c}^T x \ \textit{ subject to } \ \underline{A}x \leq \overline{b},\ x \geq 0,$$
$$\overline{f} = \min \overline{c}^T x \ \textit{ subject to } \ \overline{A}x \leq \underline{b},\ x \geq 0.$$

Optimal value range

Algorithm (Optimal value range $[\underline{f}, \overline{f}]$)

1. Compute

$$\underline{f} := \inf\ c_c^T x - c_\Delta^T |x| \ \text{ subject to } \ x \in \mathcal{M},$$

   where $\mathcal{M}$ is the primal solution set.
2. If $\underline{f} = \infty$, then set $\overline{f} := \infty$ and stop.
3. Compute

$$\overline{\varphi} := \sup\ b_c^T y + b_\Delta^T |y| \ \text{ subject to } \ y \in \mathcal{N},$$

   where $\mathcal{N}$ is the dual solution set.
4. If $\overline{\varphi} = \infty$, then set $\overline{f} := \infty$ and stop.
5. If the primal problem is strongly feasible, then set $\overline{f} := \overline{\varphi}$; otherwise set $\overline{f} := \infty$.

Optimal solution set

The optimal solution set

Denote by $\mathcal{S}(A, b, c)$ the set of optimal solutions to

$$\min c^T x \quad \text{subject to} \quad Ax = b, \; x \geq 0,$$

Then the optimal solution set is defined

$$\mathcal{S} := \bigcup_{A \in \mathbf{A},\, b \in \mathbf{b},\, c \in \mathbf{c}} \mathcal{S}(A, b, c).$$

Goal

Find a tight enclosure to $\mathcal{S}$.

Basis stability

Definition

The interval linear programming problem

$$\min \mathbf{c}^T x \quad \text{subject to} \quad \mathbf{A}x = \mathbf{b},\ x \geq 0,$$

is B-stable if B is an optimal basis for each realization.

Theorem

B-stability implies that the optimal value bounds are

$$\underline{f} = \min\ \underline{c}_B^T x \quad \textit{subject to} \quad \underline{A}_B x_B \leq \overline{b},\ -\overline{A}_B x_B \leq -\underline{b},\ x_B \geq 0,$$
$$\overline{f} = \max\ \overline{c}_B^T x \quad \textit{subject to} \quad \underline{A}_B x_B \leq \overline{b},\ -\overline{A}_B x_B \leq -\underline{b},\ x_B \geq 0.$$

Under the unique B-stability, the set of all optimal solutions reads

$$\underline{A}_B x_B \leq \overline{b},\ -\overline{A}_B x_B \leq -\underline{b},\ x_B \geq 0,\ x_N = 0.$$

Basis stability

Non-interval case

Basis B is optimal iff

C1. A_B is non-singular;

C2. $A_B^{-1}b \geq 0$;

C3. $c_N^T - c_B^T A_B^{-1} A_N \geq 0^T$.

Interval case

The problem is B-stable iff C1–C3 holds for each $A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in \mathbf{c}$.

Condition C1

- C1 says that $\mathbf{A}_B$ is regular;
- co-NP-hard problem;
- sufficient condition: $\rho\left(|((A_c)_B)^{-1}|(A_\Delta)_B\right) < 1$.

Basis stability

Non-interval case

Basis B is optimal iff

C1. A_B is non-singular;

C2. $A_B^{-1}b \geq 0$;

C3. $c_N^T - c_B^T A_B^{-1} A_N \geq 0^T$.

Interval case

The problem is B-stable iff C1–C3 holds for each $A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in \mathbf{c}$.

Condition C2

- C2 says that the solution set to $\mathbf{A}_B x_B = \mathbf{b}$ lies in $\mathbb{R}^n_+$;
- polynomial problem under assumption C1;
- sufficient condition: check of some enclosure to $\mathbf{A}_B x_B = \mathbf{b}$.

Basis stability

Non-interval case

Basis B is optimal iff

C1. A_B is non-singular;

C2. $A_B^{-1} b \geq 0$;

C3. $c_N^T - c_B^T A_B^{-1} A_N \geq 0^T$.

Interval case

The problem is B-stable iff C1–C3 holds for each $A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in \mathbf{c}$.

Condition C3

- C2 says that $\mathbf{A}_N^T y \leq \mathbf{c}_N,\ \mathbf{A}_B^T y = \mathbf{c}_B$ is strongly feasible;
- co-NP-hard problem;
- <u>sufficient</u> condition:
  $(\mathbf{A}_N^T)\mathbf{y} \leq \underline{c}_N$, where $\mathbf{y}$ is an enclosure to $\mathbf{A}_B^T y = \mathbf{c}_B$.

Basis stability

Theorem

Condition C3 holds true if and only if for each $q \in \{\pm 1\}^m$ the polyhedral set described by

$$
\begin{aligned}
((A_c)_B^T - (A_\Delta)_B^T \operatorname{diag}(q))y &\le \overline{c}_B,\\
-((A_c)_B^T + (A_\Delta)_B^T \operatorname{diag}(q))y &\le -\underline{c}_B,\\
\operatorname{diag}(q)\, y &\ge 0
\end{aligned}
$$

lies inside the polyhedral set

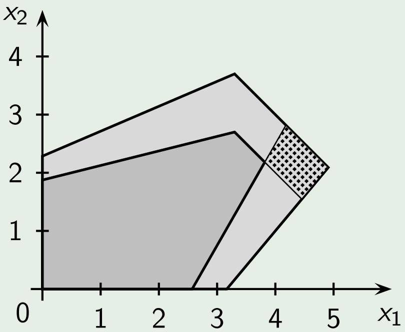
$$((A_c)_N^T + (A_\Delta)_N^T \operatorname{diag}(q))y \le \underline{c}_N,\ \operatorname{diag}(q)\, y \ge 0.$$

Example

Example

Consider an interval linear program

$$\max\ ([5,6],[1,2])^T x \ \text{ s.t. } \begin{pmatrix} -[2,3] & [7,8] \\ [6,7] & -[4,5] \\ 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} [15,16] \\ [18,19] \\ [6,7] \end{pmatrix},\ x \geq 0.$$



- union of all feasible sets in light gray,
- intersection of all feasible sets in dark gray,
- set of optimal solutions in dotted area

Basis stability – interval right-hand side

Interval case

Basis B is optimal iff

C1. A_B is non-singular;

C2. $A_B^{-1}b \geq 0$ for each $b \in \mathbf{b}$.

C3. $c_N^T - c_B^T A_B^{-1} A_N \geq 0^T$.

Condition C1

- C1 and C3 are trivial
- C2 is simplified to

$$\underline{A_B^{-1}\mathbf{b}} \geq 0,$$

  which is easily verified by interval arithmetic
- overall complexity: polynomial

Basis stability – interval objective function

Interval case

Basis B is optimal iff

C1. A_B is non-singular;

C2. $A_B^{-1}b \geq 0$;

C3. $c_N^T - c_B^T A_B^{-1} A_N \geq 0^T$ for each $c \in \mathbf{c}$

Condition C1

- C1 and C2 are trivial
- C3 is simplified to

$$A_N^T y \leq \mathbf{c}_N, \; A_B^T y = \mathbf{c}_B$$

or,

$$\overline{(A_N^T A_B^{-T})\mathbf{c}_B} \leq \underline{c}_N.$$

- overall complexity: polynomial

Conclusion

Open problems

- A sufficient and necessary condition for weak unboundedness, strong boundedness and weak optimality.
- A method for determining the image of the optimal value function.
- A sufficient and necessary condition for duality gap to be zero for each realization.
- A method to test if a basis B is optimal for some realization.
- Tight enclosure to the optimal solution set.