Algorithmic techniques for sparse graphs

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Goal

Design efficient algorithms

- polynomial-time
- approximation
- FPT
- …

for hard problems, when restricted to *sparse graphs*.

What are sparse graphs?

- whatever turns out to be useful
- generally tend to have few edges
- often bounded expansion or nowhere-dense

Properties of (some) sparse graphs

- structural decompositions
- obstructions to tree-width
- small separators
- “almost” bounded tree-width
- quasi-wideness
- generalizations of degeneracy

Structural decompositions: Example

Theorem (Robertson and Seymour)

For every H there exists k such that if H is not a minor of G, then there exist graphs $G_1, \ldots, G_n$ and sets $S_i \subseteq V(G_i)$ (apex vertices) such that

- *G can be obtained from $G_1, \ldots, G_n$ by clique-sums,*
- *$|S_i| \le k$,*
- *$G_i - S_i$ is embedded with at most k vortices of depth at most k in a surface Σ_i such that H cannot be drawn in Σ_i.*

Related results

Strengthenings in special cases:

- H has one crossing: only planar pieces without vortices or apex vertices, and pieces of bounded size (Demaine, Hajiaghayi and Thilikos)
- H is apex: apex vertices only attach to quasivortices (Demaine, Hajiaghayi and Kawarabayashi)

Generalizations:

- odd minors: pieces may also be arbitrary bipartite graphs (Demaine, Hajiaghayi and Kawarabayashi)
- topological minors: pieces may be bounded degree graphs (Grohe, Kawarabayashi, Marx and Wollan)

Other settings: perfect graphs, claw-free graphs, ...

Applications

- implies other properties
- direct algorithms; e.g., additive approximation for chromatic number
  - $\mathrm{OPT} + k - 2$ for K_k-minor-free (Demaine, Hajiaghayi and Kawarabayashi)
  - $\mathrm{OPT} + 2$ for H-minor-free, where H is apex (Demaine, Hajiaghayi and Kawarabayashi)

Tree-width and its uses

Theorem

Every problem can be solved in linear time for graphs with tree-width bounded by a constant, unless it cannot.

Theorem (Courcelle)

Any problem expressible in Monadic Second-Order Logic can be solved in linear time for graphs with tree-width bounded by a constant.

Obstructions to tree-width

Theorem

There exists f such that if $tw(G) > f(k)$, then G contains $k \times k$ wall as a topological minor.

- f exists (Robertson and Seymour)
- $f(k) \leq 400^{k^5}$ (Robertson, Seymour and Thomas)
- if G avoids a fixed minor, then f is linear (Demaine and Hajiaghayi)
- unless G contains a big clique minor, the wall is flat
- under further assumptions, grid-like graphs can be obtained only by contractions

Basic idea

- Either tree-width is small (and we can solve the problem), or
- we have a big wall (and obtain a contradiction, or it can be reduced, or . . .)

Example: crossing number is FPT (Grohe)

Does G have crossing number at most k?

- if $\mathrm{tw}(G)$ is small, then solvable in linear time (expressible in MSOL)
- if G contains a big clique minor, then its crossing number is greater than k
- if G contains a big flat wall, then we find a vertex v such that $\mathrm{cr}(G - v) \leq k$ iff $\mathrm{cr}(G) \leq k$.

Example: FPT for dominating set in graphs of bounded genus

Does G (embedded in a fixed surface Σ) contain a dominating set of size at most k?

Let $t = 3\sqrt{k} + 2$.

- if $\mathrm{tw}(G) \le f(t)$, then solvable in linear time
- otherwise, G can be contracted to a $t \times t$ partially triangulated grid and a single apex attaching to its boundary $\Rightarrow$ no dominating set of size at most k.

Bidimensional properties

Definition

A property is *bidimensional* if

- non-increasing on contractions (and possibly edge/vertex deletions)
- unbounded for “grid-like” graphs
- can be determined in polynomial-time for graphs of bounded tree-width

Consequences of bidimensionality

- FPT on appropriate classes of graphs (cf. “grid-like”)
- with some additional assumptions, PTAS’s

Separators in planar graphs

Definition

(A, B) is a *separator* in G if $G = A \cup B$, $E(A) \cap E(B) = \emptyset$ and $|V(A)|, |V(B)| \geq |V(G)/3$. Its *order* is $|V(A) \cap V(B)|$.

Theorem (Lipton and Tarjan)

Every planar graph on n vertices has a separator of order $O(\sqrt{n})$.

Generalizations

- K_k-minor free graphs have separators of order $O(\sqrt{n})$ (Alon, Seymour and Thomas)
- graph classes with subexponential expansion have sublinear separators (Plotkin and Rao; Nešetřil and Ossona de Mendez).

Applications

- Enumeration: if $\mathcal{G}$ has separators of order $O(n/\log^2 n)$, then it contains only $2^{O(n)}n!$ labelled graphs on n vertices (D. and Norine)
- Approximation:
  - separators of order $O(n^{1-\varepsilon})$ and degeneracy imply PTAS for independent set
  - PTAS for bidimensional problems with further assumptions (good behavior with respect to separators)
- Subexponential algorithms: independent set, chromatic number, ...

Neighborhoods in planar graphs

Theorem (Robertson and Seymour)

A planar graph of radius r has tree-width $O(r)$.

Corollary

If G is planar and $v \in V(G)$, the subgraph of G induced by vertices in distance at most r from v has tree-width $O(r)$.

$L_{m,k}(v)$ ...the set of vertices in distance $m \pmod{k}$ from v

Corollary

For every k, m, a planar graph G and $v \in V(G)$, the tree-width of $G - L_{m,k}(v)$ is $O(k)$.

Locally bounded tree-width

Definition

A class of graphs $\mathcal{G}$ has *locally bounded tree-width* if there exists f such that for every $G \in \mathcal{G}$, $v \in V(G)$ and $r > 0$, the subgraph of G induced by vertices in distance at most r from v has tree-width at most $f(r)$.

Examples:

- bounded maximum degree
- minor-closed classes avoiding an apex graph (Eppstein)

Applications of locally bounded tree-width

Theorem (Frick and Grohe)

For every $\varepsilon > 0$, any problem expressible in First Order Logic can be solved in $O(n^{1+\varepsilon})$ for any class of graphs with locally bounded tree-width.

Example: Does G have a dominating set of size at most k?

- find a maximal set S of vertices in pairwise distance at least three.
- if $|S| > k$, then the answer is no
- otherwise, radius of each component of G is $O(k)$, and G has bounded tree-width.

Bounded tree-width covers

Definition

A class $\mathcal{G}$ has *bounded tree-width covers* if there exists f such that for every $G \in \mathcal{G}$ and $k > 0$, there exists a partition $V(G) = V_1 \dot\cup \ldots \dot\cup V_k$ such that $\mathrm{tw}(G - V_i) \le f(k)$ for $1 \le i \le k$.

- locally bounded tree-width + minor-closed $\Rightarrow$ bounded tree-width cover.
- implies bounded expansion, sublinear separators
- holds for proper minor-closed classes (Demaine, Hajiaghayi and Kawarabayashi)
- proper minor-closed classes have also the analogical property for contractions (Demaine, Hajiaghayi and Mohar)

Applications of bounded tree-width covers

- factor 2 approximation for chromatic number
- PTAS's for many problems
  - implies FPT
- subexponential algorithms

Example: PTAS for largest independent set

Suppose that $V(G) = V_1 \dot\cup \ldots \dot\cup V_k$, and let S be an independent set in G of size $\alpha(G)$.

- for $1 \le i \le k$, we have $\alpha(G - V_i) \le \alpha(G)$
- there exists $i \in \{1, \ldots, k\}$ such that $|S \cap V_i| \le |S|/k$.

Therefore, $(1 - 1/k)\alpha(G) \le \max_{1 \le i \le k} \alpha(G - V_i) \le \alpha(G)$.

Open Problem

Problem

Characterize classes of graphs that have bounded tree-width covers.

Or, for the fractional version (there exist sets $V_1, \ldots, V_n$, such that each vertex is in at most n/k of them, and $G - V_i$ has bounded tree-width for $1 \le i \le n$)?

Big scattered sets

Definition

A (d,r)-*width* of G is the maximum size of a set A such that the distance between every two vertices of A in $G-S$ is at least d, for some set $S \subseteq V(G)$ of size at most r.

Definition

A class of graphs $\mathcal{G}$ is *quasi-wide* if there exists f such that for each d and m, only finitely many graphs in $\mathcal{G}$ have $(d, f(d))$-width at most m.

Quasi-wide classes

- bounded maximum degree $\Rightarrow$ quasi-wide, with $f(d) = 0$
- K_k-minor-free classes are quasi-wide, with $f(d) = k - 1$ (Atserias, Dawar and Kolaitis)
- hereditary graph class is quasi-wide iff it is nowhere dense (Nešetřil and Ossona de Mendez)

Applications: FPT for domination number (and variations).

Degeneracy (coloring number)

Definition

A graph G is *d-degenerate* if every subgraph of G contains a vertex of degree at most d.

Equivalently,

Definition

A graph G is *d-degenerate* if there exists a linear ordering of $V(G)$ such that every vertex has at most d neighbors before it in the ordering.

Coloring number $\mathrm{col}(G) = d + 1$, where d is the smallest such that G is d-degenerate.

Generalizations

Given a linear ordering $<$ of $V(G)$ and vertices $u < v$,

- u is *weakly k-reachable from v* if there exists a path P between u and v of length at most k, whose internal vertices are $> u$,
- u is *k-reachable from v* if the internal vertices are $> v$
- the *k-backconnectivity of v* is the maximum number of disjoint ($\leq k$)-paths from v to vertices $< v$.

Let

- weak k-coloring number $\mathrm{wcol}_k(G, <) = 1 + \max_{v \in V(G)} |\{\text{vertices weakly } k\text{-reachable from } v\}|$
- k-coloring number $\mathrm{col}_k(G, <) = 1 + \max_{v \in V(G)} |\{\text{vertices } k\text{-reachable from } v\}|$
- k-admissibility $\mathrm{adm}_k(G, <) = \max_{v \in V(G)}$ k-backconnectivity of v

Define $\mathrm{wcol}_k(G)$, $\mathrm{col}_k(G)$ and $\mathrm{adm}_k(G)$ as minimum over all linear orderings $<$ of $V(G)$.

Properties of generalized coloring numbers

- $\mathrm{adm}_k(G) \le \mathrm{col}_k(G) \le \mathrm{wcol}_k(G) \le (\mathrm{adm}_k(G)+1)^{k^2}$
- $\mathrm{col}_2(G)$ bounds acyclic chromatic number
- $\mathrm{wcol}_2(G)$ bounds star chromatic number
- bounded $\mathrm{col}_2(G) \Rightarrow$ linear Ramsey number (Chen and Schelp)
- for a class of graphs $\mathcal{G}$, $\mathrm{col}_k(\mathcal{G})$ is bounded for every k iff $\mathcal{G}$ has bounded expansion

Determining generalized coloring numbers

- $\mathrm{adm}_k(G) \le t$ can be tested in $O(n^{kt+2})$
- $\mathrm{adm}_k(G)$ can be approximated within factor of k
- in a class of graphs with bounded expansion, $\mathrm{adm}_k(G)$ can be determined in linear time

Problem

Can $\mathrm{col}_k(G)$ and $\mathrm{wcol}_k(G)$ be determined exactly, or at least approximated within constant factor, in polynomial time?

Application: approximation of domination number

Theorem

Given an ordering $<$ of vertices of G with $\mathrm{wcol}_2(G) \le c$, one can find in linear time

- *a dominating set D and*
- *a set A of vertices in pairwise distance at least three,*

such that $|D| \le c^2|A|$.

Observation: every dominating set in G has size at least $|A|$, thus $|D| \le c^2\mathrm{OPT}$.

Low tree-depth colorings

Tomorrow.