Homogeneous structures

(lecture notes)

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July 10, 2019

1 Relational structures

This follows (a simplified form) of model-theoretic structures [4]. They will be useful for us to say some results more generally and to discuss the structural Ramsey theory.

A *language* L is a set of relational symbols $R \in L$, each associated with natural number $a(R)$ called *arity.* A *(relational) L-structure* $\mathbf{A}$ is a pair $(A, (R_\mathbf{A}; R \in L))$ where $R_\mathbf{A} \subseteq A^{a(R)}$ (i.e. $R_\mathbf{A}$ is a $a(R)$-ary relation on A). The set A is called the *vertex set* or the *domain* of $\mathbf{A}$ and elements of A are *vertices.* The language is usually fixed and understood from the context (and it is in most cases denoted by L). If set A is finite we call $\mathbf{A}$ *finite structure.* We consider only structures with countably many vertices. The class of all (countable) relational L-structures will be denoted by $\mathrm{Rel}(L)$.

A *homomorphism* $f : \mathbf{A} \to \mathbf{B} = (B, (R_\mathbf{B}; R \in L))$ is a mapping $f : A \to B$ satisfying for every $R \in L$ the implication

$$(x_1, x_2, \ldots, x_{a(R)}) \in R_\mathbf{A} \implies (f(x_1), f(x_2), \ldots, f(x_{a(R)})) \in R_\mathbf{B}.$$

(For a subset $A' \subseteq A$ we denote by $f(A')$ the set $\{f(x); x \in A'\}$ and by $f(\mathbf{A})$ the homomorphic image of a structure.)

If f is injective, then f is called a *monomorphism.* A monomorphism is called *embedding* if the above implication is equivalence, i.e. if for every $R \in L$ we have

$$(x_1, x_2, \ldots, x_{a(R)}) \in R_\mathbf{A} \iff (f(x_1), f(x_2), \ldots, f(x_{a(R)})) \in R_\mathbf{B}$$

If f is an embedding which is an inclusion then $\mathbf{A}$ is a *substructure* (or *subobject*) of $\mathbf{B}$. For an embedding $f : \mathbf{A} \to \mathbf{B}$ we say that $\mathbf{A}$ is *isomorphic* to $f(\mathbf{A})$ and $f(\mathbf{A})$ is also called a *copy* of $\mathbf{A}$ in $\mathbf{B}$. Isomorphism $f : \mathbf{A} \to \mathbf{A}$ is also called an *automorphism* of $\mathbf{A}$.

Example. The following special cases of relational structures will be interesting for us:

1. Graphs: the language L_G of graphs consist of single relational symbol E of arity 2 representing edges. L_G-structure $\mathbf{A}$ is *graph* if $E_\mathbf{A}$ is symmetric and antireflexive (no loops allowed).

2. Digraphs use the same language as graphs but every L_G-structure is also a digraph (we allow directed edges, bidirecitonal edges and loops)

3. Orders: the language L_O if orders consist of single relaitonal symbol $\leq$ of arity 2. L_G-structure $\mathbf{A}$ is an *partial order* if $\leq_\mathbf{A}$ is reflexive, antisymetric and transitive. It is *linear order* if in addition it holds that for every $a, b \in A$ either $(a, b) \in \leq_\mathbf{A}$ or $(b, a) \in \leq_s trA$.

4. n-uniform hyperraphs: Language L_n of hypergraphs has single relational symbol E of arity n. L_n-structure $\mathbf{A}$ is *hypergraph* if for every $(a_1, a_2, \ldots, a_n) \in E_\mathbf{A}$ and every permutation π of $\{1, 2, \ldots, n\}$ it holds that $(a_{\pi(1)}, a_{\pi(2)}, \ldots, s_{\pi(n)}) \in E_\mathbf{A}$.

2 Homogeneity

The following definition was introduced by Fraïssé [1, 2], see also [6].

Definition 2.1 (Homogeneous structure)**.** An L-structure $\mathbf{A}$ is *homogeneous* if every isomorphism of its finite substructures extends to an automorphism of $\mathbf{A}$.

Example. All homogeneous graphs are known: finite graphs was classified by Gardiner [3] and infinite by Lachlan and Woodrow [5]. The proof is long, but the result is easy to state. Graph G is homogeneous if it is one of the following:

1. the Random graph R,
2. universal and homogeneous K_k-free graph for some $k \geq 3$,
3. a disjoint union of complete graphs, all of the same size,
4. complements of the above.
5. Two special finite cases: A 5-cycle and the "window" graph.

We also discussed that the order of rationals $(\mathbb{Q}, \leq)$ is a homogeneous and universal linear order.

Next time I will show a general method of constructing homogeneous structure and we will learn that thre are many of them.

Excercises.

1. Show that the Petersen graph is not homogeneous.
2. Consider random digraph R'. Can you formulate an extension property for digraphs? Show that R' is univeral, homogeneous and up to isomorphism unique.
3. Does the random digraph R' have the property that if its vertices are partitioned to two parts one of them is isomorphic to R'?
4. What is extension property of linear orders?
5. Modify the construction of Rado and build the homogeneous and universal triangle-free graph.

3 Fraïssé theorem

Definition 3.1. The *age*, denoted by $\mathrm{Age}(\mathbf{A})$, of a relational structure $\mathbf{A}$ is the class of all finite structures which embedds to $\mathbf{A}$. We say that $\mathbf{A}$ is *younger* then $\mathbf{B}$ if $\mathrm{Age}(\mathbf{A}) \subseteq \mathrm{Age}(\mathbf{B})$.

Definition 3.2. Let $\mathbf{A}$, $\mathbf{B}_1$ and $\mathbf{B}_2$ be L-structures and $\alpha_1 \colon \mathbf{A} \to \mathbf{B}_1$ and $\alpha_2 \colon \mathbf{A} \to \mathbf{B}_2$ embeddings. Then every L-structure $\mathbf{C}$ with embeddings $\beta_1 \colon \mathbf{B}_1 \to \mathbf{C}$ and $\beta_2 \colon \mathbf{B}_2 \to \mathbf{C}$ such that $\beta_1 \circ \alpha_1 = \beta_2 \circ \alpha_2$ is called an *amalgamation* of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$ with respect to α_1 and α_2 (see Figure 1). We will call $\mathbf{C}$ simply an *amalgamation* of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$ (as in the most cases α_1 and α_2 can be chosen to be inclusion embeddings).

Example. The following can be easily checked to be amalgamation classes:

1. The class of all finite graphs.
2. The class of all finite graphs omitting clique of size k for given $k \geq 3$
3. The class of all finite linear orders.

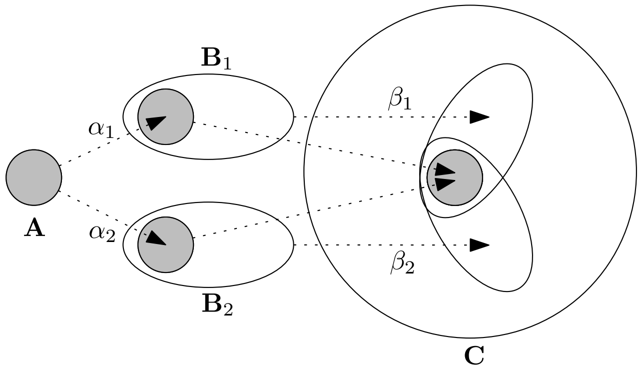

Figure 1: An amalgamation of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$.

4. The class of all finite partial orders.

5. The class of all finite metric spaces

Definition 3.3. An *amalgamation class* is an isomorphism-closed class $\mathcal{K}$ of finite L-structures satisfying the following three conditions:

1. **Hereditary property:** For every $\mathbf{A} \in \mathcal{K}$ and a substructure $\mathbf{B}$ of $\mathbf{A}$ we have $\mathbf{B} \in \mathcal{K}$;

2. **Amalgamation property:** For $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{K}$ and α_1 embedding of $\mathbf{A}$ into $\mathbf{B}_1$, α_2 embedding of $\mathbf{A}$ into $\mathbf{B}_2$, there is $\mathbf{C} \in \mathcal{K}$ which is an amalgamation of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$ with respect to α_1 and α_2.

3. There are only countably many mutually non-isomorphic structures.

Note that in the class I used the fact that Joint embedding is a special case of amalgamation property.

Now we are able to state the beautiful Fraïssé theorem:

Theorem 3.1 (Fraïssé [1])**.**

(a) A class $\mathcal{K}$ of finite L-structures is the age of a countable homogeneous structure $\mathbf{M}$ if and only if $\mathcal{K}$ is the amalgamation class.

(b) If the conditions of (a) are satisfied, then the structure $\mathbf{M}$ is unique up to isomorphism

It is easy to check that the age of a countable homogeneous structure is an amalgamation class. The way to construct $\mathbf{M}$ given its age is the following concept:

Definition 3.4. Suppose $\mathcal{K}$ is a class of finite L-structures. An increasing chain

$$A_0 \subseteq A_1 \subseteq A_2 \leq A_3 \subseteq \cdots$$

of structures in $\mathcal{K}$ is called a *rich sequence* if:

1. for all $A \in \mathcal{K}$ there is some $i < \omega$ and a embedding $e : A \to A_i$;

2. for all $B \in \mathcal{K}$, $f : A_j \to B$ there is $k \geq j$ and a $g : B \to A_k$ such that $g(f(a)) = a$ for all $a \in A_j$.

A *Fraïssé limit* of $\mathcal{K}$ is an L-structure which is the union of a rich sequence of substructures.

It is not hard to show that for amalgamation class rich sequence exists (both conditions are special cases of amalgamation property).

The Fraïssé limit has the *extension property*: for every $\mathbf{A} \subseteq \mathbf{B} \in \mathcal{K}$ and embedding $e : \mathbf{A} \to \mathbf{M}$ there exists embedding $f : \mathbf{B} \to \mathbf{M}$ extending e. With this it is possible to use back-and-forth the same way as for the Random graph to obtain homogeneity and uniqueness of $\mathbf{N}$.

Excercises. Show that for every homogeneous structure it holds that every countable structure $\mathbf{M}'$ such that $\mathrm{Age}(\mathbf{M}') \subseteq \mathrm{Age}\,\mathbf{M}$ has embedding to $\mathbf{M}$. In other words $\mathbf{M}$ is *universal* for all structures of the same age as $\mathbf{M}$ or younger.

References

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