Outline of Topics

Structures

Λ-algebras

Λ a set, S a partial semigroup, and X a set

A **Λ-algebra over S based on X** is an assignment to each $\lambda \in \Lambda$ of a function from a subset of X to S such that for $s_0, \dots, s_k \in S$ and $\lambda_0, \dots, \lambda_k \in \Lambda$ there exists $x \in X$ with $s_0\lambda_0(x), \dots, s_k\lambda_k(x)$ defined.

A Λ-algebra over A based on X is **total** if A a semigroup and the domain each $\lambda \in \Lambda$ is equal to X.

A Λ-algebra is **point based** if it is total and X consist of one point, usually denoted by $\bullet$.

$\mathcal{A}$ and $\mathcal{B}$ are total Λ-algebras with $\mathcal{A}$ being over A and based on X and $\mathcal{B}$ being over B and based on Y.

A **homomorphism from $\mathcal{A}$ to $\mathcal{B}$** is a pair of functions f, g such that $f\colon X \to Y$, $g\colon A \to B$, g is a homomorphism of semigroups, and, for each $x \in X$ and $\lambda \in \Lambda$, we have

$$\lambda(f(x)) = g(\lambda(x)).$$

A total Λ-algebra from a Λ-algebra—following Bergelson, Blass, Hindman

$\mathcal{S}$ a Λ-algebra over S based on X

γX is the set of all ultrafilters $\mathcal{V}$ on X such that for $s \in S$ and $\lambda \in \Lambda$

$$\{x \in X \colon s\lambda(x) \text{ is defined}\} \in \mathcal{V}.$$

γS is the set of all ultrafilters $\mathcal{U}$ on S such that for $s \in S$

$$\{t \in S \colon st \text{ is defined}\} \in \mathcal{U}.$$

γS is a semigroup with convolution: $(\mathcal{U}, \mathcal{V}) \to \mathcal{U} * \mathcal{V}$, where

$$C \in \mathcal{U} * \mathcal{V} \iff \{s \in S \colon \{t \in S \colon st \in C\} \in \mathcal{V}\} \in \mathcal{U}.$$

In other words,

$$C \in \mathcal{U} * \mathcal{V} \iff \forall^{\mathcal{U}} s \, \forall^{\mathcal{V}} t \, (st \in C).$$

Each λ induces a function from γX to γS by the formula

$$C \in \lambda(\mathcal{V}) \text{ iff } \lambda^{-1}(C) \in \mathcal{V}.$$

This procedure gives a **total** Λ**-algebra** $\gamma\mathcal{S}$ **over** γS **based on** γX.

Theorem

Assume we have a Λ-algebra over S and based on X.

A sequence (x_n) of elements of X is **basic** if for all $n_0 < \cdots < n_l$ and $\lambda_0, \ldots, \lambda_l \in \Lambda$

$$\lambda_0(x_{n_0})\lambda_1(x_{n_1})\cdots\lambda_l(x_{n_l}) \tag{1}$$

is defined in S.

Assume we additionally have a point based Λ-algebra $\mathcal{A}$ over A.

A coloring of S is **$\mathcal{A}$-tame on (x_n)** if the color of elements in (1) depends only on

$$\lambda_0(\bullet)\lambda_1(\bullet)\cdots\lambda_l(\bullet) \in A$$

provided $\lambda_k(\bullet)\cdots\lambda_l(\bullet) \in \Lambda(\bullet)$ for all $k \leq l$.

Theorem (S.)

Fix a finite set Λ. Let $\mathcal{S}$ be a Λ-algebra over S, and let $\mathcal{A}$ be a point based Λ-algebra. Let $(f, g)\colon \mathcal{A} \to \gamma\mathcal{S}$ be a homomorphism.

Then for each $D \in f(\bullet)$ and each finite coloring of S, there exists a basic sequence (x_n) of elements of D on which the coloring is $\mathcal{A}$-tame.

The goal: **produce homomorphisms from point based algebras $\mathcal{A}$ to $\gamma\mathcal{S}$**

Tensor products

Fix a partial semigroup S.

Λ_0, Λ_1 finite sets

$\mathcal{S}_i$, for $i = 0, 1$, Λ_i-algebras over S with $\mathcal{S}_i$ is based on X_i

Put

$$\Lambda_0 \star \Lambda_1 = \Lambda_0 \cup \Lambda_1 \cup (\Lambda_0 \times \Lambda_1).$$

Define

$$\mathcal{S}_0 \otimes \mathcal{S}_1$$

to be a $\Lambda_0 \star \Lambda_1$-algebra over S based on $X_0 \times X_1$ as follows: with

$$\lambda_0,\ \lambda_1,\ (\lambda_0, \lambda_1) \in \Lambda_0 \star \Lambda_1$$

associate partial functions $X_0 \times X_1 \to S$ by letting

$$\begin{aligned} \lambda_0(x_0, x_1) &= \lambda_0(x_0), \\ \lambda_1(x_0, x_1) &= \lambda_1(x_1), \\ (\lambda_0, \lambda_1)(x_0, x_1) &= \lambda_0(x_0)\lambda_1(x_1). \end{aligned}$$

Proposition (S.)

Fix semigroups A and B. For $i = 0, 1$, let $\mathcal{A}_i$ and $\mathcal{B}_i$ be Λ_i-algebras over A and B, respectively. Let

$$(f_0, g)\colon \mathcal{A}_0 \to \mathcal{B}_0 \ \text{ and } \ (f_1, g)\colon \mathcal{A}_1 \to \mathcal{B}_1$$

be homomorphisms. Then

$$(f_0 \times f_1, g)\colon \mathcal{A}_0 \otimes \mathcal{A}_1 \to \mathcal{B}_0 \otimes \mathcal{B}_1$$

is a homomorphism.

Let $\mathcal{S}_i$, $i = 0, 1$, be Λ_i-algebras over S based on X_i. Consider

$$\gamma\mathcal{S}_0 \otimes \gamma\mathcal{S}_1 \textbf{ and } \gamma(\mathcal{S}_0 \otimes \mathcal{S}_1).$$

Both are $\Lambda_0 \star \Lambda_1$-algebras over γS.

The first one is based on $\gamma X_0 \times \gamma X_1$, the second one on $\gamma(X_0 \times X_1)$.

There is a natural map $\gamma X_0 \times \gamma X_1 \to \gamma(X_0 \times X_1)$ given by

$$(\mathcal{U}, \mathcal{V}) \to \mathcal{U} \times \mathcal{V},$$

where, for $C \subseteq X_0 \times X_1$,

$$C \in \mathcal{U} \times \mathcal{V} \Longleftrightarrow \{x_0 \in X_0 \colon \{x_1 \in X_1 \colon (x_0, x_1) \in C\} \in \mathcal{V}\} \in \mathcal{U}.$$

Proposition (S.)

Let S be a partial semigroup. Let $\mathcal{S}_i$, $i = 0, 1$, be Λ_i-algebras over S. Then

$$(f, \mathrm{id}_{\gamma S})\colon \gamma\mathcal{S}_0 \otimes \gamma\mathcal{S}_1 \to \gamma(\mathcal{S}_0 \otimes \mathcal{S}_1),$$

where $f(\mathcal{U}, \mathcal{V}) = \mathcal{U} \times \mathcal{V}$, is a homomorphism.

$\mathcal{A}$ a point based Λ-algebra over a semigroup A

Fix a natural number $r > 0$.

Let

$$\Lambda_{<r}(\bullet) = \{\lambda_0(\bullet) \cdots \lambda_m(\bullet) \colon m < r,\ \lambda_i \in \Lambda \text{ for } i \leq m\}.$$

Note

$$\Lambda(\bullet) \subseteq \Lambda_{<r}(\bullet) \subseteq A.$$

$\mathcal{S}$ a Λ-algebra over a partial semigroup S
(x_n) a basic sequence in $\mathcal{S}$

A coloring of S is r-$\mathcal{A}$-**tame on** (x_n) if the color of elements of the form

$$\lambda_0(x_{n_0})\lambda_1(x_{n_1})\cdots\lambda_l(x_{n_l}),$$

for $n_0 < \cdots < n_l$ and $\lambda_0, \ldots, \lambda_l \in \Lambda$ depends only on

$$\lambda_0(\bullet)\lambda_1(\bullet)\cdots\lambda_l(\bullet) \in A$$

provided

$$\lambda_k(\bullet)\cdots\lambda_l(\bullet) \in \Lambda_{<r}(\bullet) \text{ for all } k \le l.$$

The following corollary of the theorem is its generalization.

Corollary

Fix a finite set Λ and a natural number r. Let $\mathcal{S}$ be a Λ-algebra, $\mathcal{A}$ a point based Λ-algebra, and $(f, g)\colon \mathcal{A} \to \gamma\mathcal{S}$ a homomorphism.
Then for each $D \in f(\bullet)$ and each finite coloring of S, there exists a basic sequences (x_n) of elements of D on which the coloring is r-$\mathcal{A}$-tame.