Lectures 2+

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3 An extension of Fraïssé's Theorem

We give a generalization of Fraïssé 's Theorem. Further generalizations are possible (though the basic structure of the proof is always the same). The exposition is based on my Bonn notes and some notes of Martin Ziegler (see his webpage[1]).

We shall work with a class $\mathcal{K}$ of finite L-structures and a distinguished class of substructures $A \leq B$, pronounced 'A is a *strong* substructure of B' (the terminology is not standard). If $B \in \mathcal{K}$, then an embedding $f : A \to B$ is a $\leq$-embedding if $f(A) \leq B$. We shall assume that $\leq$ satisfies:

(N1) If $B \in \mathcal{K}$ then $B \leq B$ (so isomorphisms are $\leq$-embeddings);

(N2) If $A \leq B \leq C$ (and $A, B, C \in \mathcal{K}$), then $A \leq C$ (so if $f : A \to B$ and $g : B \to C$ are $\leq$-embeddings, then $g \circ f : A \to C$ is a $\leq$-embedding).

In this case, we say that $(\mathcal{K}; \leq)$ is a *strong class* of finite structures.

Example 3.1. Let $\mathcal{K}$ be the class of finite 2-oriented digraphs. If $A \subseteq B \in \mathcal{K}$ write $A \sqsubseteq B$ to mean that A is successor-closed in B. Then $(\mathcal{K}; \sqsubseteq)$ is a strong class.

Definition 3.2. We say that strong class $(\mathcal{K}; \leq)$ has the *amalgamation property* (for strong embeddings) if whenever A_0, A_1, A_2 are in $\mathcal{K}$ and $f_1 : A_0 \to A_1$ and $f_2 : A_0 \to A_2$ are strong, there is $B \in \mathcal{K}$ and strong $g_i : A_i \to B$ (for $i = 1, 2$) with $g_1 \circ f_1 = g_2 \circ f_2$. The class has the *joint embedding property* if for all $A_1, A_2 \in \mathcal{K}$ there is some $C \in \mathcal{K}$ and strong $f_1 : A_1 \to C$ and $f_2 : A_2 \to C$.

If $\emptyset \in \mathcal{K}$ and $\emptyset \leq A$ for all $A \in \mathcal{K}$, then the JEP is a special case of the AP. Note that $(\mathcal{K}; \sqsubseteq)$ in Example 3.1 has the amalgamation property (and $(\mathcal{K}; \subseteq)$ does not have the AP).

Definition 3.3. Suppose $(\mathcal{K}; \leq)$ is a strong class of finite L-structures. An increasing chain

$$A_0 \leq A_1 \leq A_2 \leq A_3 \leq \ldots$$

is called a *rich sequence* if:

[1]'Strong Fraisse limits' `http://home.mathematik.uni-freiburg.de/ziegler/preprints/starker_fraisse.pdf`

1. for all $A \in \mathcal{K}$ there is some $i < \omega$ and a strong embedding $A \to A_i$;
2. for all strong $f : A_i \to B$ there is $j \geq i$ and a strong $g : B \to A_j$ such that $g(f(a)) = a$ for all $a \in A_i$.

A *Fraïssé limit* of $(\mathcal{K}; \leq)$ is an L-structure which is the union of a rich sequence of substructures.

We want to talk about strong substructures of a Fraïssé limit. There are several ways of doing this, but we follow Ziegler's method. In the examples which we will be working with, there will be a more natural way of describing this which extends the notion of strong substructure from the finite case.

Definition 3.4. Suppose M is a Fraïssé limit of a strong class $(\mathcal{K}; \leq)$. If A is a finite substructure of M, say that $A \leq M$ if M is the union of a rich sequence of substructures which starts with A.

Note that this property is preserved by automorphisms of M.

Lemma 3.5. *Suppose $(\mathcal{K}; \leq)$ is a strong class with the amalgamation property and M is a Fraïssé limit. If $A \leq A_0 \leq M$, then $A \leq M$.*

Proof. Suppose $A_0 \leq A_1 \leq \cdots$ is a rich sequence with union M. To show that $A \leq A_0 \leq A_1 \ldots$ is a rich sequence, let $f : A \to B \in \mathcal{K}$ be strong. Use the AP to find strong $h : A_0 \to C$ and $g : B \to C$ with $h(a) = g(f(a))$ for $a \in A$. Now apply richness of the A_i-sequence to h. □

Theorem 3.6. *Suppose $(\mathcal{K}; \leq)$ is a strong class of finite structures with AP and JEP. Then*

1. *rich sequences for the class, and therefore Fraïssé limits, exist.*
2. *Suppose M, M' are Fraïssé limits of the class. If $A \leq M$ and $A' \leq M'$ are finite and $f : A \to A'$ is an isomorphism, then f extends to an isomorphism $g : M \to M'$.*

Proof. (1) Build the rich sequence $A_0 \leq A_1 \leq \ldots$ inductively ensuring that:

- if $C \in \mathcal{K}$ there is an i and a $\leq$-embedding $f : C \to A_i$;
- if $f : A_i \to B \in \mathcal{K}$ is strong then there is $j \geq i$ and a $\leq$-embedding $g : B \to A_j$ with $g(a) = a$ for all $a \in A$.

To perform tasks of the first type, we use JEP; for the second type we can use AP. There are only countably many tasks to perform, so we can arrange that all are completed during the construction of the A_i.

(2) We extend f by a back-and-forth argument in the usual way. The key point is to show how to extend the domain of f ('forth'): the 'back' step is symmetrical.

So let $A \leq A_1 \leq A_2 \leq A_3 \leq \ldots$ and $A' \leq A_1 \leq A'_2 \leq A'_3 \leq \ldots$ be rich sequences with Fraïssé limits M and M' respectively. It will suffice to find an embedding $h : A_1 \to M'$ which extends f and has $h(A_1) \leq M'$. By the second condition in the definition of richness, there is a strong embedding $h : A_1 \to A'_j$ extending f, for some j. By the Lemma, $h(A_1) \leq M'$. $\square$

NOTE: In the above, if M is a Fraïssé limit of the strong class $(\mathcal{K}; \leq)$ (with AP and JEP), then any isomorphism between finite $\leq$-substructures of M extends to an automorphism of M. We refer to this property as $\leq$-*homogeneity* of M.

In Example 3.1, if M is a Fraïssé limit, then $A \sqsubseteq M$ just means that A is successor-closed in M. More generally, we have:

Remarks 3.7. Suppose that $(\mathcal{K}; \leq)$ is a strong class with AP and that:

(N3) whenever $A \leq B$ and $C \in \mathcal{K}$ is a substructure of B which contains A, then $A \leq C$.

If M is a Fraïssé limit with rich sequence $(A_i)_{i<\omega}$, then, for a finite $A \subseteq A_i$, we have $A \leq M$ iff $A \leq A_i$. The condition implies that if $A \leq C \in \mathcal{K}$ and $A \subseteq B \leq C$, then $A \leq B$.

The Fraïssé limit in Example 3.1 is not ω-categorical. We finish this section by giving a condition which guarantees ω-categoricity.

Remarks 3.8. Suppose $(\mathcal{K}; \leq)$ in Theorem 3.6 has only finitely many isomorphism types of structure of each finite size. Suppose also that there is a function $F : \mathbb{N} \to \mathbb{N}$ such that if $B \in \mathcal{K}$ and $A \subseteq B$ with $|A| \leq n$, then there is $C \leq B$ with $A \subseteq C$ and $|C| \leq F(n)$. Then the Fraïssé limit M is ω-categorical.

To see this we note that $\mathrm{Aut}(M)$ has finitely many orbits on M^n. Indeed, by $\leq$-homogeneity there are finitely many orbits on $\{\bar{c} \in M^{F(n)} : \bar{c} \leq M\}$ and any $\bar{a} \in M^n$ can be extended to an element of this set.

4 Hrushovski's predimension construction

All of this is due to Ehud Hrushovski. There are several variations on the construction. We shall focus on the the version which produces an ω-categorical 2-sparse graph (with vertices of infinite valency).

4.1 Predimensions

Notation 4.1. Let L be a language for graphs: so we have a single 2-ary relation symbol R. We work with the class $\mathcal{C}$ of finite L-structures A where R is symmetric and irreflexive. Thus, the set R^A of instances of R in A can be thought of as a set of 2-subsets from A, so $R^A \subseteq [A]^2$. Define the *predimension* of A to be:

$$\delta(A) = 2|A| - |R^A|.$$

Remarks 4.2. This can be done more generally. Note that a graph $\Gamma = (B; R)$ is 2-very sparse iff $\delta(A) \geq 0$ for all finite $A \subseteq B$.

Lemma 4.3. *(Submodularity) Suppose $A \in \mathcal{C}$ and $B, C \subseteq A$. Then*

$$\delta(B \cup C) \leq \delta(B) + \delta(C) - \delta(B \cap C).$$

Moreover, there is equality here iff $R^{B\cup C} = R^B \cup R^C$ (that is, B are freely amalgamated over $B \cap C$ in A).

Proof. Note that

$$\begin{aligned} &\delta(B) + \delta(C) - \delta(B \cap C) - \delta(B \cup C) \\ &= -(|R^B| + |R^C| - |R^{B\cap C}| - |R^{B\cup C}|) \\ &= |R^{B\cup C}| - (|R^B| + |R^C| - |R^B \cap R^C|) \\ &= |R^{B\cup C}| - |R^B \cup R^C| \geq 0 \end{aligned}$$

with equality iff $R^{B\cup C} = R^B \cup R^C$. □

Definition 4.4. Suppose $B \in \mathcal{C}$ and $A \subseteq B$. Write $A \leq_d B$ if $\delta(A) < \delta(B')$ for all B' with $A \subset B' \subseteq B$ and say that A is *d-closed* in B.

Lemma 4.5. *Let $B \in \mathcal{C}$.*

1. *If $A \leq_d B$ and $X \subseteq B$, then $A \cap X \leq_d X$.*
2. *If $A \leq_d C \leq_d B$, then $A \leq_d B$.*
3. *If $A_1, A_2 \leq_d B$, then $A_1 \cap A_2 \leq_d B$.*

Proof. (1) Let $Y \subseteq X$ with $A \cap X \subset Y$. Note that $Y \cap A = X \cap A$. Then by submodularity:

$$\delta(A \cup Y) \leq \delta(A) + \delta(Y) - \delta(Y \cap A) \text{ so } \delta(A \cup Y) \leq \delta(A) + \delta(Y) - \delta(X \cap A).$$

Therefore

$$\delta(Y) - \delta(A \cap X) \geq \delta(A \cup Y) - \delta(A).$$

As $A \leq_d B$, this is > 0.

(2) Let $A \subset X \subseteq_{fin} B$. By (1), $C \cap X \leq_d X$ so

$$\delta(A) \leq \delta(X \cap C) \leq \delta(X)$$

(the first of these coming from $A \leq_d C$ and the second from $X \cap C \leq_d X$). As $A \subset X$, at least one of these inequalities is strict.

(3) By (1) $A_1 \cap A_2 \leq_d A_1$, so the result follows from (2). □

Remarks 4.6. (3) shows that if $A \subseteq B$ and $S = \{A_1 : A \subseteq A_1 \leq_d B\}$, then $\bigcap S \leq_d B$. So there is a smallest $\leq_d$- subset of B which contains A: denote it by $\mathrm{cl}_B^d(A)$. It is easy to see that cl_B^d is a closure operations on B.

Lemma 4.7. *For $A \subseteq B \in \mathcal{C}$ we have $\delta(A) \geq \delta(\mathrm{cl}_B^d(A))$.*

Proof. Amongst all the subsets X of B containing A, consider the ones for which $\delta(X)$ is as small as possible. Amongst these, choose one, C, with as many elements as possible. Clearly $\delta(C) \geq \delta(A)$ and if $C \subset D \subseteq B$, then $\delta(C) < \delta(D)$. So $C \leq_d B$ and therefore $A \subseteq \mathrm{cl}_B^d(A) \leq_d C \leq_d B$. By choice of C we have $\delta(C) \leq \delta(\mathrm{cl}_B^d(A))$, therefore $C = \mathrm{cl}_B^d(A)$. The result follows. □

From Lemma 4.5, the notions of distinguished substructure $\leq_d$ satisfies (N1), (N2) and (N3). However, we do not have the JEP. For example, suppose $B_1, B_2 \in \mathcal{C}$ are finite and $\delta(B_2) < 0$. Let C be the free amalgam (disjoint union) of B_1 and B_2. Then $\delta(B_1) + \delta(B_2) < \delta(B_1)$ so $B_1 \not\leq_s C$. So it makes sense to exclude structures of negative predimension. For an ω-categorical structure, we also want to ensure that the d-closure of a finite subset of a structure in the class we consider is bounded uniformly in the size of the subset. This is the point of the following definition.

Definition 4.8. Let $F : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ be a continuous, increasing function with $F(x) \to \infty$ as $x \to \infty$, and $F(0) = 0$. Let

$$\mathcal{C}_F = \{B \in \mathcal{C} : \delta(A) \geq F(|A|) \text{ for all } A \subseteq B\}.$$

Theorem 4.9. *1. If $B \in \mathcal{C}_F$ and $A \subseteq B$ then*

$$|\mathrm{cl}_B^d(A)| \leq F^{-1}(\alpha|A|).$$

2. If $(\mathcal{C}_F, \leq_d)$ is an amalgamation class, then the generic structure M_F is ω-categorical.

Proof. (1) By Lemma 4.7 we have $\delta(\mathrm{cl}^d_B(A)) \leq \delta(A) \leq 2|A|$. Thus (by definition of $\mathcal{C}_F$) we have $|\mathrm{cl}^d_B(A)| \leq F^{-1}(\alpha|A|)$.

(2) This follows from Remarks 3.8.

□

Main Construction: Let F as in Definition 4.8 be such that:

- F is piecewise smooth;
- the right derivative F' is non-increasing;
- $F'(x) \leq 1/x$ for all $x > 0$.

The we claim that $(\mathcal{C}_F, \leq_d)$ is a free amalgamation class.

Indeed, suppose $A \leq_d B_1, B_2 \in \mathcal{C}_F$ and let E be the free amalgam of B_1 and B_2 over A. We need to show that $E \in \mathcal{C}_F$. Clearly we may assume $A \neq B_i$.

Suppose $X \subseteq E$. We need to show that $\delta(X) \geq F(|X|)$. Now, X is the free amalgam over $A \cap X$ of $B_1 \cap X$ and $B_2 \cap X$ and $A \cap X \leq_d B_i \cap X$ (by Lemma 4.5(1)). So we can assume $X = E$ and check that $\delta(E) \geq F(|E|)$.

Note that $\delta(E) = \delta(B_1) + \delta(B_2) - \delta(A)$ and $|E| = |B_1| + |B_2| - |A|$.

The effect of the conditions on F is that for $x, y \geq 0$

$$F(x + y) \leq F(x) + yF'(x) \leq F(x) + y/x.$$

We can assume that

$$\frac{\delta(B_2) - \delta(A)}{|B_2| - |A|} \geq \frac{\delta(B_1) - \delta(A)}{|B_1| - |A|}$$

and note that the latter is at least $1/|B_1|$ (as δ is integer-valued and $A \leq_d B_1$).

Then

$$\delta(E) = \delta(B_1) + (|B_2| - |A|)\frac{\delta(B_2) - \delta(A)}{|B_2| - |A|}$$

$$\geq F(|B_1|) + (|B_2| - |A|)/|B_1| \geq F(|E|)$$

(taking $x = |B_1|$ and $y = |B_2| - |A|$).

This concludes the proof of the claim.

Example 4.10. We use this to produce an example of a connected ω-categorical graph whose automorphism group is transitive on vertices and edges, and whose smallest cycle is a 5-gon.

Take

$$F(1) = 2; F(2) = 3; F(5) = 5; F(k) = \log(k) + 5 - \log(5) \text{ for } k \geq 5.$$

Then one can check that:

- The smallest cycle in $\mathcal{C}_F$ is a 5-gon;
- If $a \in A \in \mathcal{C}_F$ then $a \leq_d A$
- If $ab \subseteq B \in \mathcal{C}_F$ is an edge then $ab \leq_d B$
- $(\mathcal{C}_F, \leq_d)$ is an amalgamation class (the proof of AP in the previous example applies if at least one of B_1, B_2 has size ≥ 5; the other cases can be checked individually).
- The Fraïssé limit M_F is connected. Given non-adjacent $a, b \in M_F$ consider $A = \mathrm{cl}^d(ab)$. As $\delta(A) \leq \delta(ab) = 4$ we have $|A| \leq 3$. So either A is a path of length 2 (with endpoints a, b) or $A = ab$, so $ab \leq_d M_F$. In the latter case, consider a path B of length 3 with end points a, b. Then $ab \leq_d B$ so there is a $\leq_d$ copy of B in M_F over ab. In particular, ab are at distance 3 in M_F.

It follows that the smallest cycle in the Fraïssé limit M_F is a 5-cycle and $\mathrm{Aut}(M_F)$ is transitive on vertices and edges. In fact, the argument shows that M_F is distance transitive of diameter 3.