Combinatorics, Topology & Computing

Martin Tancer

Department of Applied Mathematics, Charles University

Three topics

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   - Higher-dimensional generalization of graph planarity.
   - Brings a lot of new combinatorial, topological or computational challenges.

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   - Higher-dimensional generalization of graph planarity.
   - Brings a lot of new combinatorial, topological or computational challenges.
2. Sphere/ball recognition
   - Here I want to emphasize combinatorial tools.
3. Algebra & topology in extremal combinatorics
   - Purely combinatorial weak saturation problems.
   - Solutions using algebraic and topological tools.

1. EMBEDDABILITY

Graph planarity

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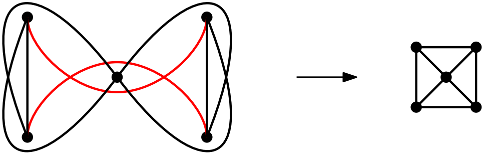
A graph is planar if and only if it does not admit a subdivision of K_5 or $K_{3,3}$.

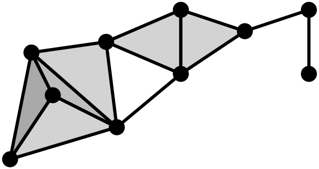
Theorem (Hanani-Tutte theorem)

A graph is planar if and only if it admits a drawing in which every pair of vertex-disjoint edges has an even number of intersections.



Higher-dimensional graph planarity = embeddability

Simplicial complex: A collection of simplices glued along faces.



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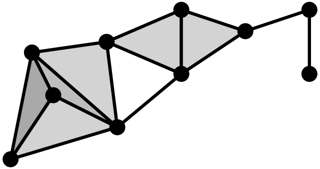
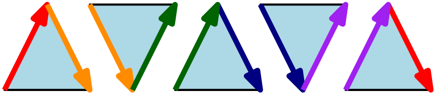


- Can we embed a given complex into $\mathbb{R}^d$?



$\hookrightarrow \mathbb{R}^3$?

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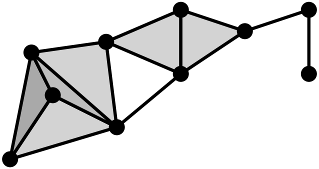

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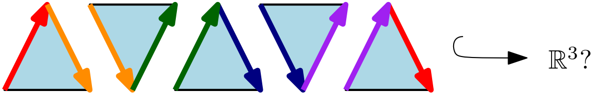
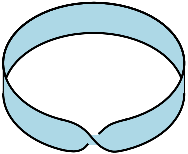
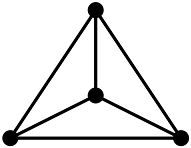
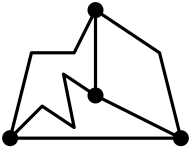




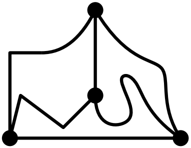
Types of embeddings



Linear



Piecewise linear
= linear on some subdivision



Topological

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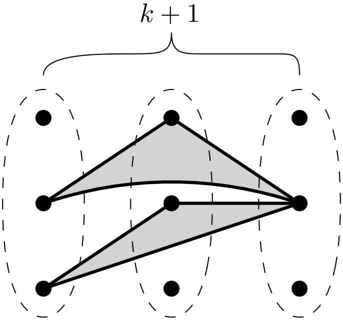

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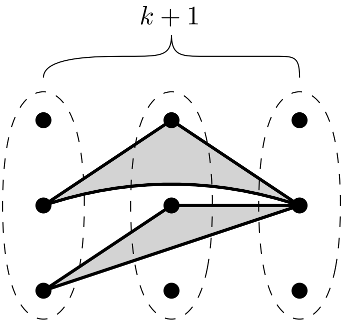

- Generalizes $K_{3,3}$. (There is also another one generalizing K_5.)

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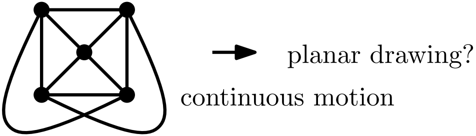

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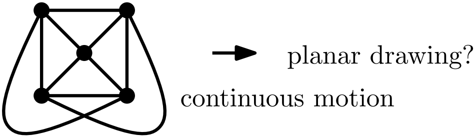

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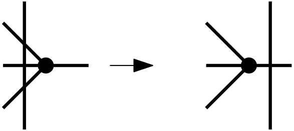

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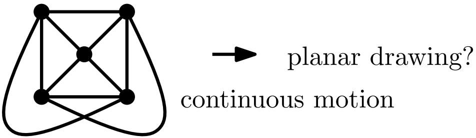

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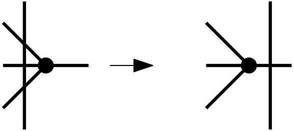

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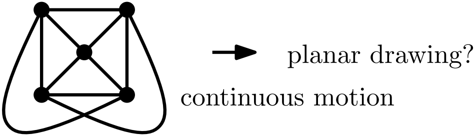

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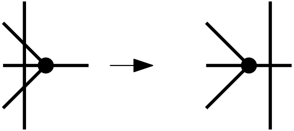
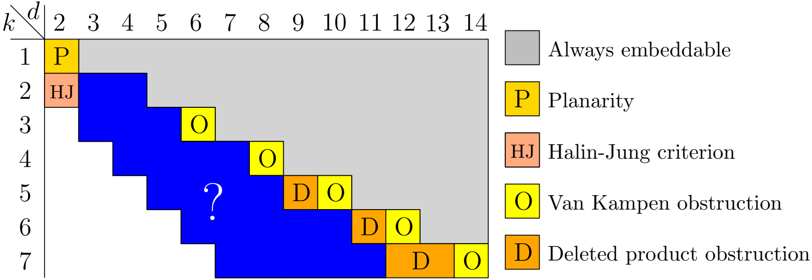


- Each change keeps the parity of the number of pairs of intersecting vertex disjoint edges.
- Can be generalized to higher dimensions as a combinatorial/algebraic criterion.

Criteria in other dimensions

- Given k and d, is there an equivalent criterion for embeddability of k-dimensional complexes in $\mathbb{R}^d$?



Algorithmic viewpoint on embeddability

Embeddability, $\text{EMB}_{k \to d}$

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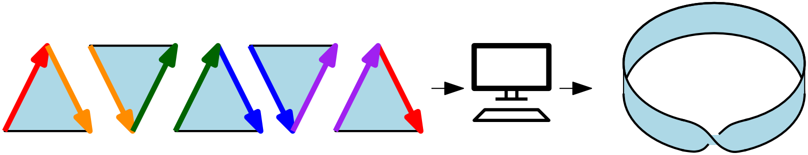
QUESTION: Does $\mathbf{K}$ embed into $\mathbb{R}^d$?



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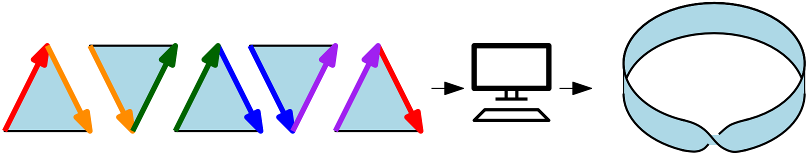
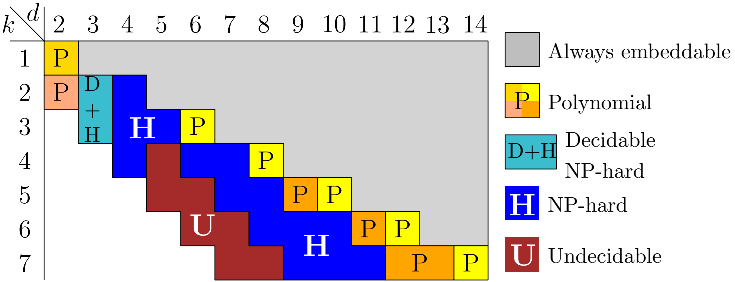


Theorem (Matoušek, T., Wagner '11)

$\mathrm{EMB}_{k\to d}$ is NP-hard if $d \geq 4$ and $k > \frac{2}{3}d - 1$. In addition $\mathrm{EMB}_{d-1\to d}$ and $\mathrm{EMB}_{d\to d}$ are undecidable.

Embeddability - current state of art



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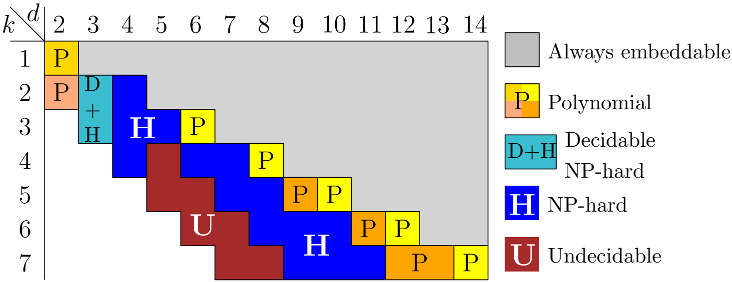

P $\mathrm{EMB}_{k\to d}$ is polynomial time solvable for $k \leq \frac{2d-3}{3}$, series of papers by Čadek, Krčál, Matoušek, Sergeraert, Vokřínek, Wagner '13.

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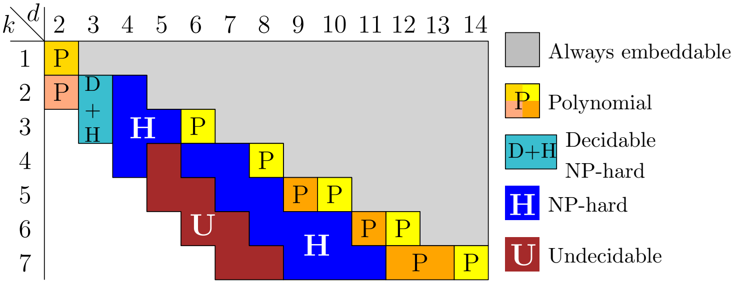

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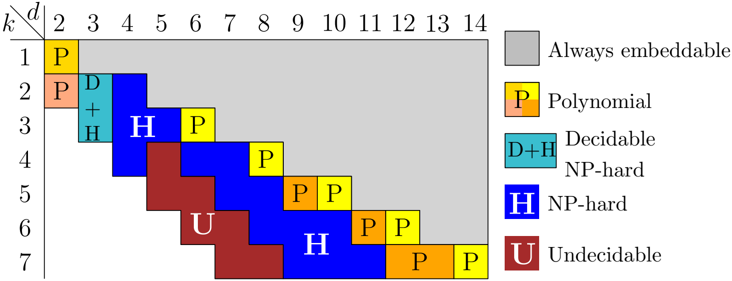
- D: $\text{EMB}_{2 \to 3}$ and $\text{EMB}_{3 \to 3}$ are decidable, Matoušek, Sedgwick, T., Wagner '14
- H: $\text{EMB}_{2 \to 3}$ and $\text{EMB}_{3 \to 3}$ are NP-hard, de Mesmay, Rieck, Sedgwick, T. '18

Some curiosities



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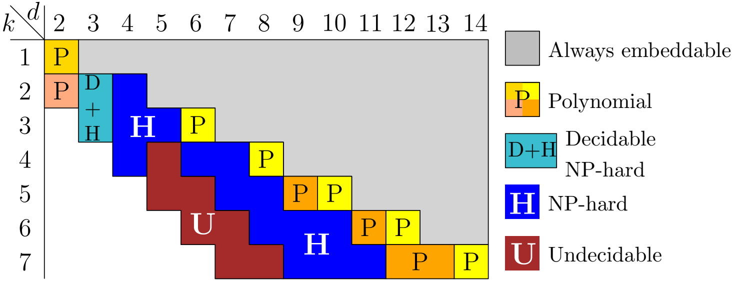

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- Skopenkov, T. '19: So called almost embeddability differs from a 'deleted product criterion' provided that P$\neq$NP. (Direct proof is not known.)

2. SPHERE/BALL RECOGNITION

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Question: Is a given triangulated topological space homeomorphic to a d-ball?

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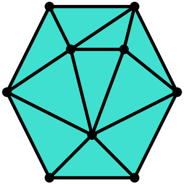

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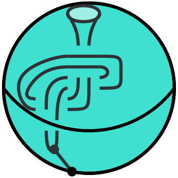

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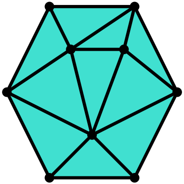

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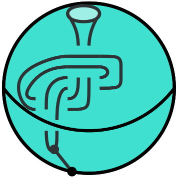

$d = 3$

- $d = 1$: trivial
- $d = 2$: easy
- $d = 3$: in NP and co-NP (modulo GRH); **polynomial unknown**
- $d = 4$: nobody knows
- $d \geq 5$: undecidable

More on 3-ball recognition

NP-membership: Ivanov '01, Schleimer '04

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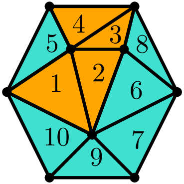
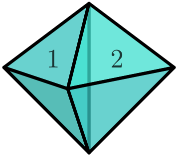
Even more optimistic: There is a combinatorial polynomial time algorithm.

Combinatorial tool: Shellability





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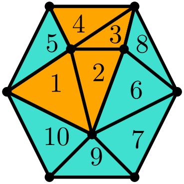

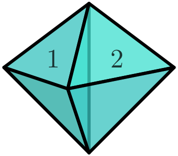

Shellability: A useful tool that allows induction.

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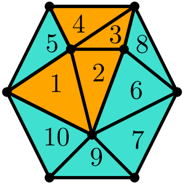

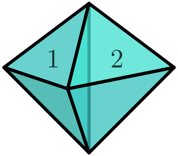

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Applications in:

- Piecewise linear topology
- Polytope theory
- Theory of partially ordered sets
- Combinatorial commutative algebra

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- Check whether the input space is a 3-manifold.
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- The high-level approach of Danaraj and Klee in principle makes sense but maybe we need some other combinatorial objects or some other tricks.

3. ALGEBRA & TOPOLOGY IN EXTREMAL COMBINATORICS

Weak Saturation

Definition

Let F, G, H be graphs with $G \subseteq F$ and $V(G) = V(H)$. We say that G is weakly H-saturated in F, if the 'remaining' edges in $E(F) \setminus E(G)$ admit an ordering $e_1, \ldots, e_k$ such that for every i the graph $G \cup \{e_1, \ldots, e_i\}$ contains a copy of H which contains e_i.

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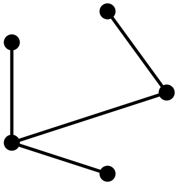
Example: Any spanning tree is weakly K_3-saturated in K_n.



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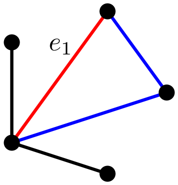

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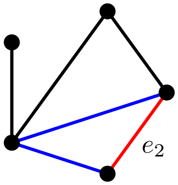

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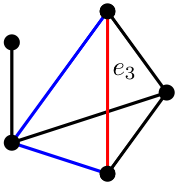

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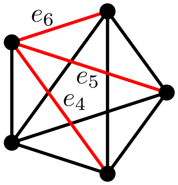

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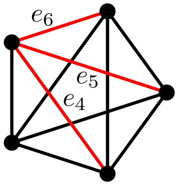

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The weak saturation number $\mathrm{wsat}(F, H)$ is the minimum number of edges in a weakly saturated graph.

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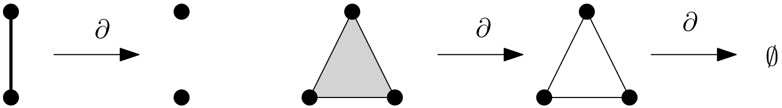

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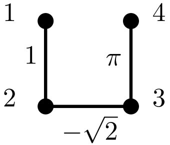

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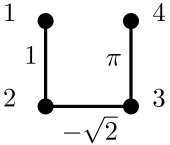

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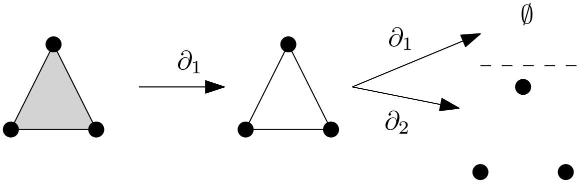
- Allows to define mutually independent boundary operators:



Classical results

Theorem (Frankl, Kalai $\sim$'82, independently)

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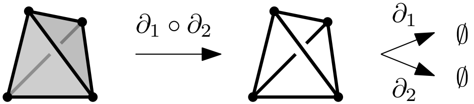

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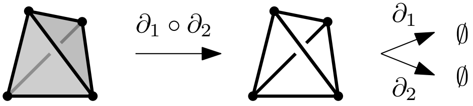

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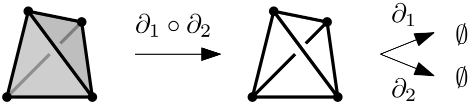

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- What about multipartite graphs?

New results—colorful setting

- Split the basis of the exterior algebra into 'color classes' $(e_1^1, \ldots, e_{n_1}^1), (e_1^2, \ldots, e_{n_2}^2), \ldots, (e_1^k, \ldots, e_{n_k}^k)$.

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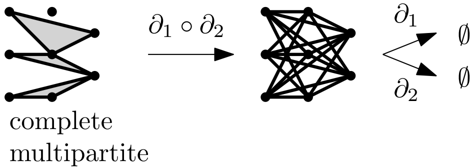

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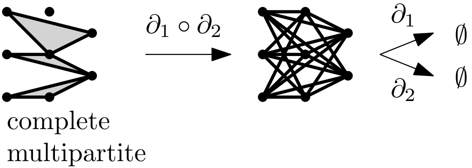

- With this, we (Bulavka, T., Tyomkyn '23) could determine:
    - $\mathrm{wsat}(K_{n_1,\ldots,n_k}, K_{r,\ldots,r})$
    - $\mathrm{wsat}(K_{n_1,\ldots,n_k}, K_{r_1,\ldots,r_k})$ (in so called directed setting)

Other applications of similar tools in extremal combinatorics

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Other (future) applications?

- As an example, Kalai and Nevo '19 proposed an approach towards Turán's $(3, 4)$-conjecture via similar tools. (Determine the minimum possible number of edges in a 3-uniform hypergraph such that every four vertices span an edge.)