Measurable Combinatorics

Oleg Pikhurko
University of Warwick

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Borel/Measurable/Descriptive Combinatorics: Typical setup

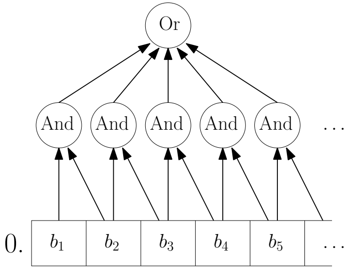
- Objects: graphs on topological spaces
  - Vertex set can be e.g. $[0, 1]$ or $\{0, 1\}^{\mathbb{N}}$
  - Degrees are uniformly bounded
- Locally checkable labelling (LCL) problem
  - Maximal independent set
  - Proper vertex / edge k-colouring
  - Perfect matching
- Aim: "Constructive" labelling

Example: irrational rotation graph $\mathcal{R}_\alpha$

- Fix irrational $\alpha \in \mathbb{R}$
- $T : [0, 1) \to [0, 1), \quad x \mapsto x + \alpha \pmod 1$
  - $\cong$ rotation of circle by angle $\frac{\alpha}{2\pi}$
- Graph $\mathcal{R}_\alpha := \Big([0, 1), \{\{x, Tx\} : x \in [0, 1)\}\Big)$
  - Every component is an infinite double ray
- Axiom of Choice: $\chi(\mathcal{R}_\alpha) = 2$
- “Constructive" 2-colouring?

Borel sets

- $\mathcal{B} = \{$Borel sets$\}$: σ-algebra generated by open sets
- $X \subseteq [0, 1]$ is Borel iff $\mathbb{I}_X$ is computed by well-founded countable Boolean circuit of binary expansion
- E.g. $\{x : \exists$ two consecutive digits 1$\} \in \mathcal{B}$



- Descriptive set theory

Measurable sets

- Fixed measure μ
- A is measurable if $\exists\, B \in \mathcal{B}$ with $\mu(A \triangle B) = 0$
  - "Definable" except on measure 0
- $\mathcal{L} := \{\text{measurable sets}\}$
- σ-algebra

Measurable chromatic number of $\mathcal{R}_\alpha$

- Irrational α
- $x \in [0, 1)$ is adjacent to $x \pm \alpha \pmod 1$
- Measurable chromatic number $\chi_{\mathcal{L}}(\mathcal{R}_\alpha) > 2$:
  - $[0, 1) = A \cup B$, measurable 2-colouring
  - $T(A) = B$, $T(B) = A$
    - Lebesgue measure $\lambda(A) = 1/2$
    - $T^2(A) = A$
  - Ergodicity of T^2: $\lambda(A) = 0$ or 1 $\Rightarrow\Leftarrow$

Borel 3-colouring of $\mathcal{R}_\alpha$

- Fix small $\varepsilon > 0$
- $V_3 := [0, \varepsilon)$ gets colour 3
- Every component of $\mathcal{R}_\alpha - A$ is a path of length $\leq N$
- Colour by the parity of the number of steps until V_3
- Each V_i is a finite union of half-open intervals
  - $Y_j := \{x : \min\{n \geq 0 : T^n(x) \in V_3\} = j\}$
    - $Y_0 = V_3$
    - $Y_j = T^{-1}(Y_{j-1}) \setminus (Y_0 \cup \cdots \cup Y_{j-1})$
  - $V_2 := Y_2 \cup Y_4 \cup Y_6 \cup \ldots$
  - $V_1 := Y_1 \cup Y_3 \cup Y_5 \cup \ldots$
- Borel chromatic number $\chi_{\mathcal{B}}(\mathcal{R}_\alpha) \leq 3$

Motivation

- Actions of a finitely generated group $\Gamma = \langle S \rangle$ on V
  - Schreier graph on V with edges $\{x, \gamma.x\}$, $\gamma \in S$
- Countable Borel equivalence relations (CBER)
  - Cost: min average degree of a spanning subgraph
- Factors of IID in probability theory
  - Independent random seeds $s(x) \in [0, 1]$ for $x \in \mathbb{Z}^d$
  - Aim: invariant labelling satisfying given constraints
  - Graph on $[0, 1]^{\mathbb{Z}^d}$ with edges corresponding to shifts
  - Timár'23: Matching of optimal tail between two Poisson processes on $\mathbb{Z}^d$ (or on $\mathbb{R}^d$)
    - Bowen-Kun-Sabok' $\geq$'23 $\Rightarrow$ desired matching
- LOCAL algorithms
- Limits of bounded degree graphs
- Equidecompositions

Connections to distributed computing

- r-LOCAL algorithm on n-vertex G
  - Deterministic: Label of x is a function of $N^{\leq r}(x)$
  - Randomised: Uniform $V \to [0, 1]$, require $\mathbb{P}[\text{fail}] < 1/n$
- Bernshteyn'23: $o(\log n)$-LOCAL algorithm
  - Deterministic $\Rightarrow$ Borel solution
  - Randomised $\Rightarrow$ measurable solution
- Vertex colouring for graphs with max degree Δ:

$\Delta + 1$ any G	Goldberg-Plotkin-Shannon'88: $O(\log^* n)$-DET	Kechris-Solecki-Todorcevic'99: Borel
Δ forest	Chang-Kopelowitz-Pettie'19: NOT $o(\log n)$-DET	Marks'16: NOT Borel
$\Delta \geq 3$ no $K_{\Delta+1}$	Ghaffari-Hirvonen-Kuhn-Maus'18: $o(\log n)$-RAND	Conley–Marks–Tucker-Drob'16: measurable

- Brandt-Chang-Grebík-Grunau-Rozhoň-Vidnyánszky'22: On trees, $O(\log n)$-DET iff Baire

Limits of bounded degree graphs

- r-sample from finite G: output rooted graph $G[N^{\leq r}(x), x]$, up to isomorphism, for uniform $x \in V$
- r-sample from Borel $\mathcal{G}$ with measure μ: take $x \sim \mu$
- $G_n \to \mathcal{G}$ locally: $\forall r$ r-samples converge
- Aldous'07, Elek'07: limit $(\mathcal{G}, \mu)$ exists
- Measure preservation: $\exists$ Borel XY-matching $\Rightarrow$ $\mu(X) = \mu(Y)$
- Aldous-Lyons'07: Is $\forall$ m.p. $\mathcal{G}$ a limit of finite graphs ?
- Global-local convergence
  - Similarity: any colouring in one graph can be approximately reproduced in the other
  - Hatami-Lovàsz-Szegedy'14: m.p. graphs as limits
  - Kun-Thom' $\geq$'23: $\exists$ m.p. $\mathcal{G}$, not limit of finite graphs

Global-local convergence

- $(\mathcal{G}, \nu)$: a limit of 3-regular random graphs
- $(\mathcal{H}, \mu)$: a limit of 3-regular random bipartite graphs
- $\forall\ \varepsilon > 0\ \ \exists\ A \subseteq V(\mathcal{H})$ st $\mu(A) = \frac{1}{2}$ and average degree of $\mathcal{H}[A]$ is $\leq \varepsilon$
- False for $(\mathcal{G}, \nu)$

Equidecompositions

- $A, B \subseteq \mathbb{R}^k$ are equidecomposable ($A \sim B$):
  - $\exists\ A = A_1 \sqcup \cdots \sqcup A_n$
  - $\exists\ B = B_1 \sqcup \cdots \sqcup B_n$

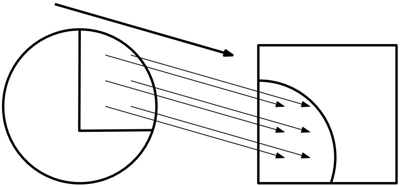
  st $\forall\, i\ \ \exists$ isometry γ_i with $B_i = \gamma_i(A_i)$
- $k \geq 3$
- Banach-Tarski Paradox'24: ball in $\mathbb{R}^k$ can be doubled
- $\Rightarrow$ Every mean (finitely-addivite isometry-invariant) $m : \{\text{bounded subsets of } \mathbb{R}^k\} \to [0, \infty)$ is 0
- Pieces cannot be measurable
- What can be done “constructively"?

Measurable pieces in $\mathbb{R}^k$, $k \geq 3$

- Necessary conditions for $A \sim_{\mathcal{L}} [0,1]^k$:
  - A is measurable and Lebesgue measure $\lambda(A) = 1$
  - finitely many copies of A cover $[0,1]^k$
  - A is bounded
- Grabowski-Máthé-P.'22: sufficient for $k \geq 3$
- Banach-Ruziewicz Problem'30: Is every mean $\{\text{bounded}\} \cap \mathcal{L} \to [0,\infty)$ a multiple of λ ?
- Margulis'82: Yes!
- Grabowski-Máthé-P.'22: new proof

Equidecompositions via graph matching

- Isometries $\gamma_1, \ldots, \gamma_n$
- Bipartite graph $\mathcal{H}$:
  - $V := A \sqcup B$
  - $E := \{xy \in A \times B : \exists\, i\ \gamma_i(x) = y\}$



- Matching $\mathcal{M} \subseteq E \iff$ disjoint $A_1, \ldots, A_n \subseteq A$ st $\gamma_1(A_1), \ldots, \gamma_n(A_n) \subseteq B$ are disjoint
- $\exists$ perfect matching $\Rightarrow A \sim B$

Borel maximal matching

- Isometries $\gamma_1, \ldots, \gamma_n$
- Bipartite graph $\mathcal{H} = (V, E)$:
  - $V = A \sqcup B$
  - $E = \{xy \in A \times B : \exists\, i \in [n]\ \gamma_i(x) = y\}$
- Greedy Algorithm:
  - $A_1 := A \cap \gamma_1^{-1}(B)$
  - $B_1 := \gamma_1(A_1)$
  - $A_2 := (A \setminus A_1) \cap \gamma_2^{-1}(B \setminus B_1)$
  - $B_2 := \gamma_2(A_2)$
  - $A_3 := (A \setminus (A_1 \cup A_2)) \cap \gamma_3^{-1}(B \setminus (B_1 \cup B_2))$
  - ...
- A_i, B_i are i-LOCAL functions of A, B
- Elek-Lippner'10: $\exists$ Borel $\mathcal{M}_i$ without augmenting paths of length $\leq 2i - 1$

Augmenting paths in finite graphs

- Partial matching $\mathcal{M}$ in bipartite finite graph H
- No augmenting path of length $\leq 2\ell + 1 \;\Rightarrow\;$ fraction of unmatched vertices is $\leq \frac{1}{\ell+1}$
- Finite bipartite graph H on $A \sqcup B$ is a *c*-expander:
  - $|A| = |B|$
  - $\forall X$ in one part $|N(X)| \geq \min\big((1+c)|X|, \frac{2}{3}|A| \big)$
- Lyons-Nazarov'11: c-expander & no augmenting ($\leq 2\ell + 1$)-path $\;\Rightarrow\;$ $|\text{unmatched in } A| \leq (1+c)^{-\ell/2}|A|$

Measurable version

- $\mathcal{H} = (A \sqcup B, E)$ is measurable c-expander:
  - $\mu(A) = \mu(B)$
  - measure preserving
  - $\forall$ Borel X in a part
    $\mu(N(X)) \geq \min\big((1+c)\mu(X), \frac{2}{3}\mu(A) \big)$
- Lyons-Nazarov'11: No augmenting path of length $\leq 2\ell + 1 \;\Rightarrow\; \mu(\text{unmatched in } A) \leq (1+c)^{-\ell/2}\mu(A)$
- Lyons-Nazarov'11: $\exists$ measurable perfect matching
  - Iteratively augment for $\ell = 0, 1, 2$...
  - Change in measure $\leq (2\ell+1)(1+c)^{-\ell/2}\mu(A)$
  - Borel-Cantelli: $\mu(\text{changes infinitely often}) = 0$

Proof of $A \sim_{\mathcal{L}} B$ in $\mathbb{R}^k$, $k \geq 3$

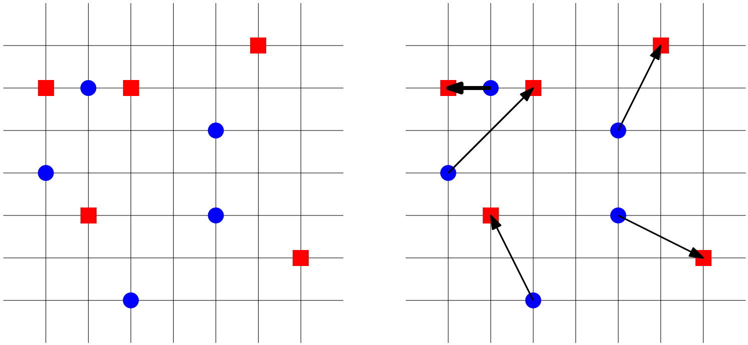
- Spectral gap for $\mathrm{SO}(3) \curvearrowright \mathbb{S}^2$ (Drinfeld'84, Lubotzky-Phillips-Sarnak'86)
- Lift to $\mathbb{R}^k$, $k \geq 3$
- $\Rightarrow \exists\, c > 0$ & finite S of isometries of $\mathbb{R}^k$ st Schreier graph is measurable c-expander on $A \times B$
- Apply Lyons-Nazarov'11

$\mathbb{R}^k$ with $k \leq 2$

- Banach'23: $A \sim B$, measurable $\Rightarrow \lambda(A) = \lambda(B)$
  - Impossible to double a disk
- Tarski's Circle Squaring Problem'25: Is a disk equidecomposable to a square ?
- Laczkovich'90: Yes!
- Grabowski-Máthé-P.'17: Measurable pieces
- Marks-Unger'17: Borel pieces
- Máthé-Noel-P. $\geq$'23: Boolean combinations of F_σ sets & Jordan measurable

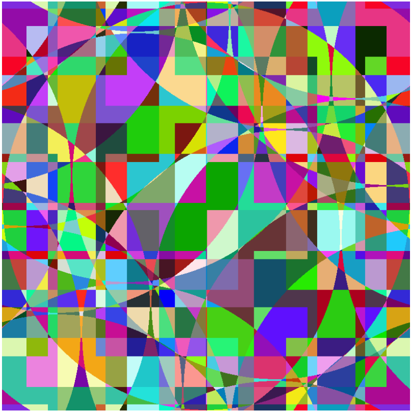
Local picture

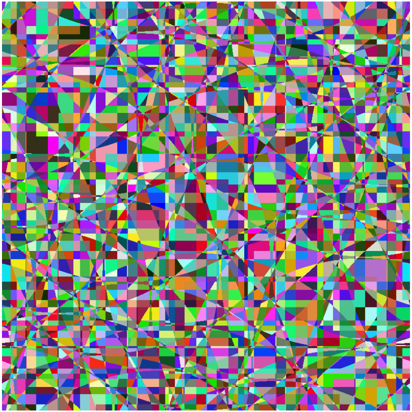
- Work in the torus $\mathbb{T}^k := \mathbb{R}^k/\mathbb{Z}^k$ (i.e. modulo 1)
- Random vectors $\mathbf{x}_1, \ldots, \mathbf{x}_d \in \mathbb{T}^k$
- Use translations only (by multiples of $\mathbf{x}_1, \ldots, \mathbf{x}_d$)



r-LOCAL functions

- r-LOCALI functions of A and B: Boolean combination of A and B, shifted by $\sum_{i=1}^{d} n_i \mathbf{x}_i$ with $\mathbf{n} \in \{-r, \ldots, r\}^d$
- Venn diagrams for $d = 2$ and $r = 1, 2$:

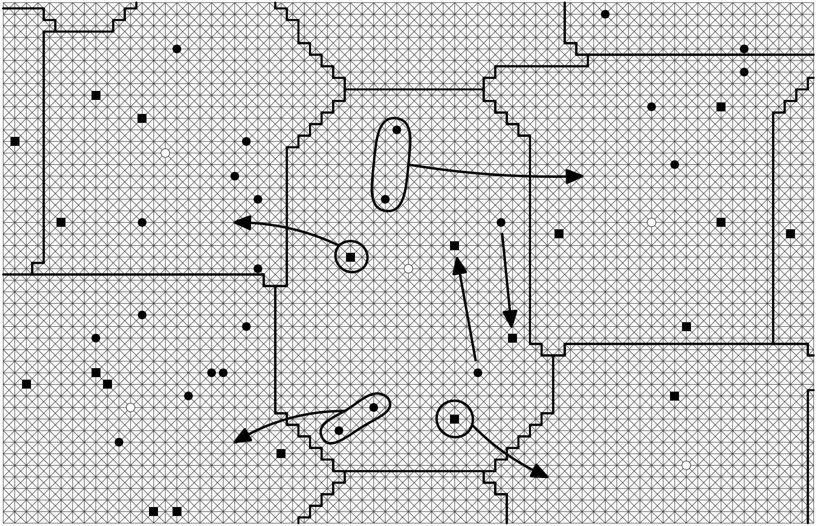




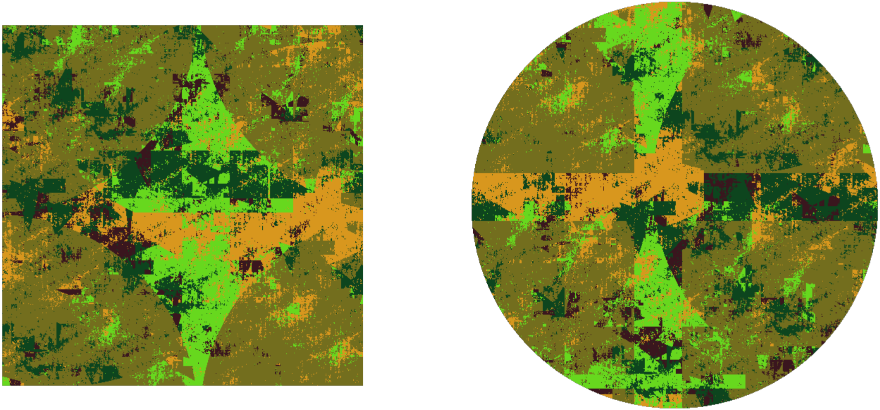
Combinatorial part

- $\mathbb{Z}^d$ with equi-distributed red and blue points
- Laczkovich'90: exists a matching that moves each vertex at distance $\leq D$
  - $\mathcal{H} := (A, B, \mathrm{dist}_{\mathbb{Z}^d} \leq D)$ satisfies Hall's condition
- Grabowski-Máthé-P.'17: $\forall$ unmatched $\mathbf{x} \in A$ and $\mathbf{y} \in B$ $\exists$ an augmenting path in $\mathcal{H}$ from $\mathbf{x}$ to $\mathbf{y}$ of length $O(\mathrm{dist}(\mathbf{x}, \mathbf{y}))$
- Marks-Unger'17, Máthé-Noel-P. $\geq$'23: construct a Borel real-valued flow and then round it

Bounded integer-valued *AB*-flow $\Rightarrow$ $A \sim B$



Discrete circle squaring (by András Máthé)



580×580 torus, 5 pieces, working modulo 1

Open problems

- Are every two bounded Borel $A, B \subseteq \mathbb{R}^3$ of equal measure and with non-empty interior equidecomposable with Borel pieces?
- Aldous-Lyons'07: Is every measure-preserving Borel graph a local limit of finite graphs?
- Bernshteyn'23: Is there a distributed computing counterpart to Borel solutions?
- "Constructive" versions of Lovász Local Lemma?
  - Csóka-Grabowski-Máthé-P.-Tyros' $\geq$'23: Borel for subexponential growth dependency graphs
  - Fischer-Ghaffari'16, Ghaffari-Harris-Kuhn'18: $o(\log n)$-RAND if $p(\Delta + 1)^8 < 2^{-15}$
  - Bernshteyn'23: measurable if $p(\Delta + 1)^8 < 2^{-15}$

Some initial pointers

- Bernshteyn: *“Descriptive Combinatorics and Distributed Algorithms”*, Notices of AMS, 2022
- Grebík: *“Problem session"* in Oberwolfach Report 8/2022 *“Descriptive Combinatorics, LOCAL Algorithms and Random Processes"*, 2022
- Kechris-Marks: *“Descriptive graph combinatorics"*, 139 pages (October 2020)
- Marks: *“Measurable graph combinatorics"*, ICM 2022
- P: *“Borel combinatorics of locally finite graphs"*, Surveys in Combinatorics, CUP 2021

Thank you!