Turán densities of tight cycles

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Joint with Nina Kamčev and Alexey Pokrovskiy

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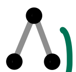
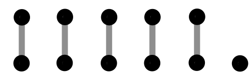
* $\mathrm{ex}(n,$  $) = \lfloor \frac{n}{2} \rfloor$.



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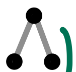
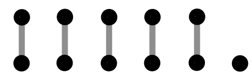
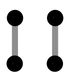
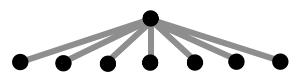
* $ex(n,$  $) = \lfloor \frac{n}{2} \rfloor$.



* $ex(n,$  $) = n-1$



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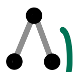
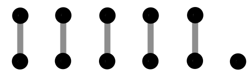
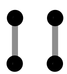
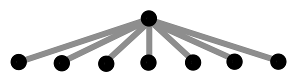
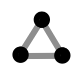
* $ex(n,$  $) = \lfloor \frac{n}{2} \rfloor$.



* $ex(n,$  $) = \begin{cases} n-1 & \text{if } n \neq 3 \\ 3 & \text{if } n = 3. \end{cases}$





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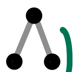
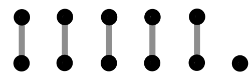
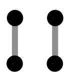
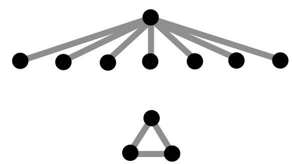
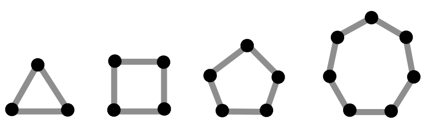
* $ex(n,$  $) = \lfloor \frac{n}{2} \rfloor$.

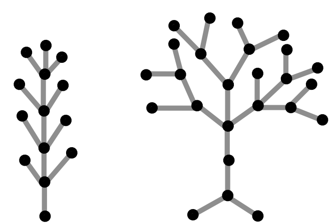


* $ex(n,$  $) = \begin{cases} n-1 & \text{if } n \neq 3 \\ 3 & \text{if } n = 3. \end{cases}$





* $ex(n, \{\text{all cycles}\}) = n-1$.



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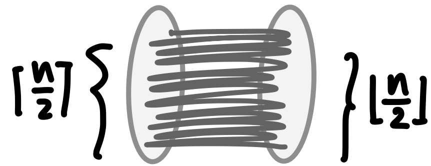

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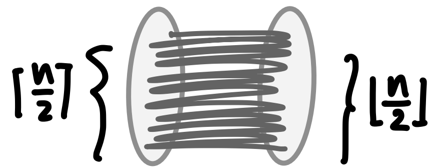
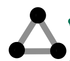


Mantel 1907: $ex(n,$  $) = \lfloor \frac{n^2}{4} \rfloor$.

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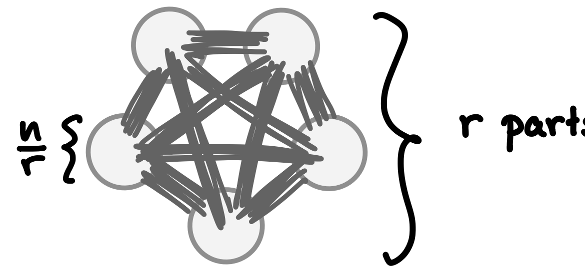

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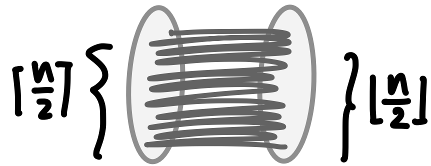

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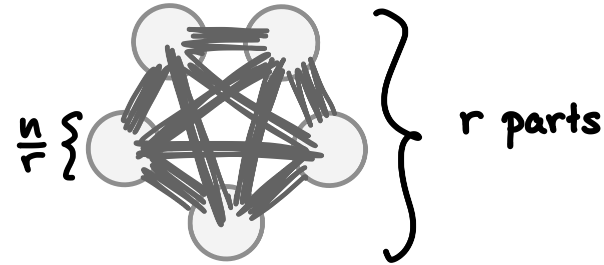

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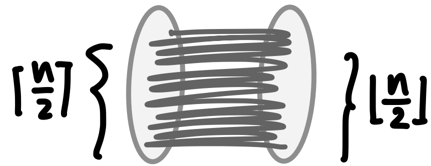

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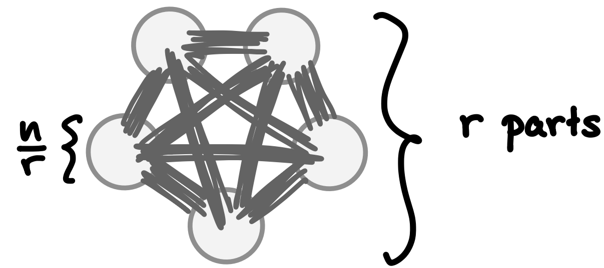

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(**Füredi–Gunderson (2013)** determined $\mathrm{ex}(n, C_{2k+1})$ for all n.)

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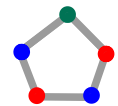

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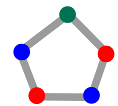

This determines $ex(n,H)$ asymptotically when $\chi(H) \geq 3$.

Turán densities of bipartite graphs

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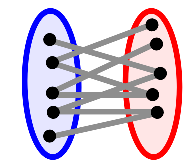

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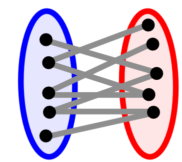

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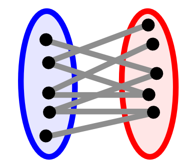

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Turán numbers and densities of hypergraphs

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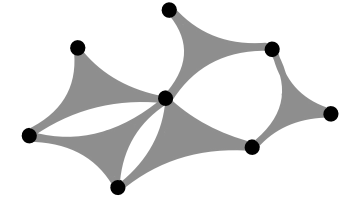

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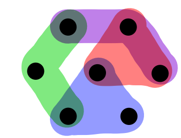

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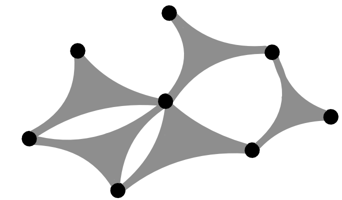

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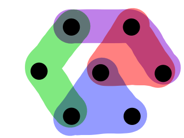

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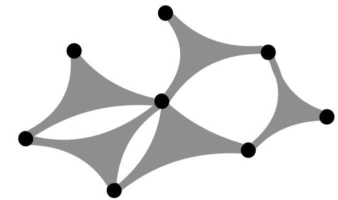

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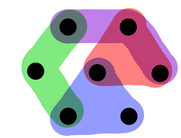

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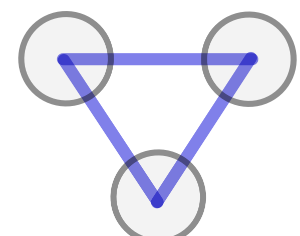

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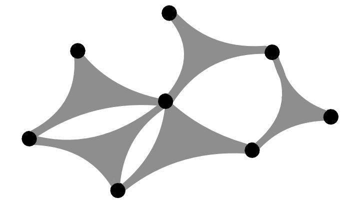

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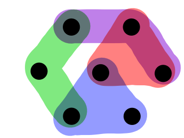

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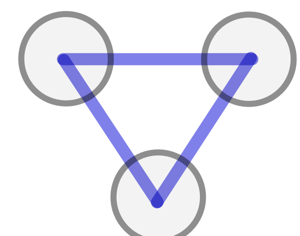

* $\pi(\mathcal{H})$ is known for very few hypergraphs $\mathcal{H}$.

Known Turán densities of hypergraphs

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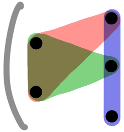
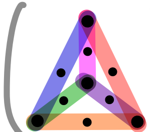
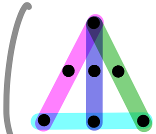
$\pi\Big($ 

$\Big) = \pi\Big($ 

$\Big) = \pi\Big($ 

$\Big) = \frac{2}{9}$

Frankl–Füredi 1983

Mubayi 2006

Pikhurko 2007

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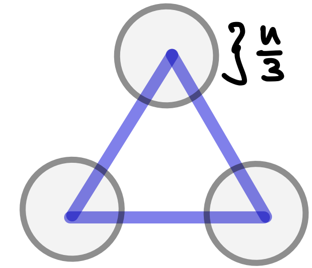
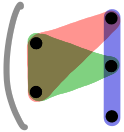
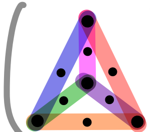
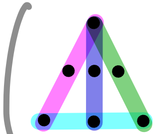
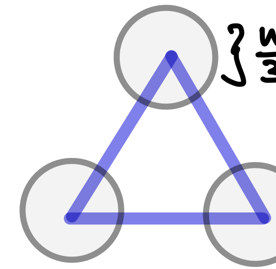


Known Turán densities of hypergraphs

F_5 — extended clique — generalised fan

$\pi\Big($  $\Big) = \pi\Big($  $\Big) = \pi\Big($  $\Big) = \frac{2}{9}$



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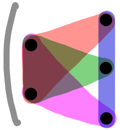
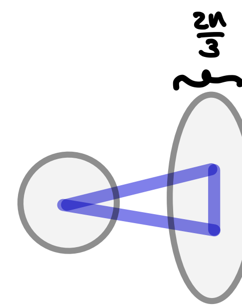
independent neighbourhoods

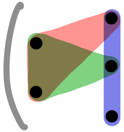
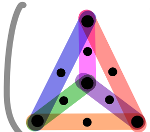
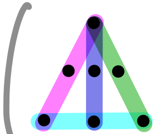
$\pi\Big($  $\Big) = \frac{4}{9}.$

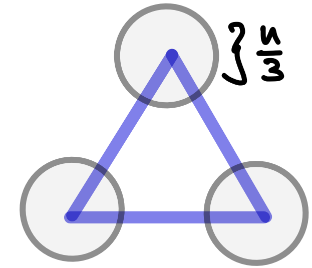


Füredi–Pikhurko–Simonovits 2003

Known Turán densities of hypergraphs

F_5 — extended clique — generalised fan

$\pi\Big($  $\Big) = \pi\Big($  $\Big) = \pi\Big($  $\Big) = \frac{2}{9}$



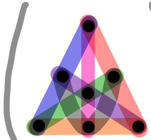
Frankl–Füredi 1983

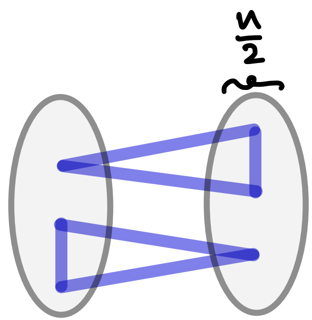
Mubayi 2006

Pikhurko 2007

Mubayi–Pikhurko 2007

Fano plane

$\pi\Big($  $\Big) = \frac{3}{4}$



de Caen–Füredi 2000

Keevash–Sudakov 2005

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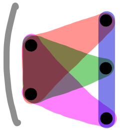
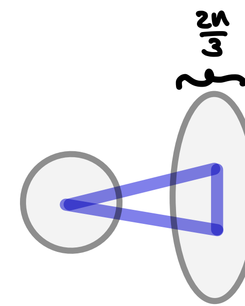
$\pi\Big($  $\Big) = \frac{4}{9}.$



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Some conjectured Turán densities

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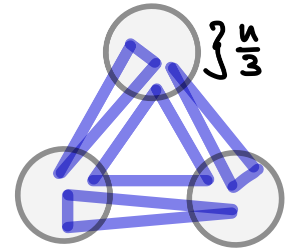

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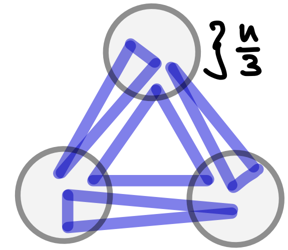

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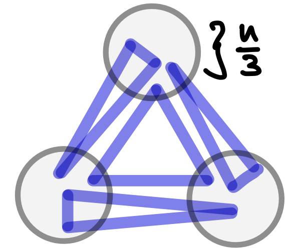

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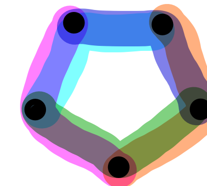

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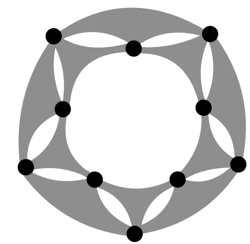

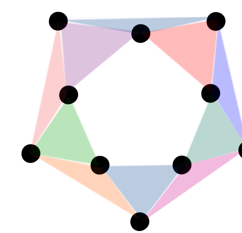

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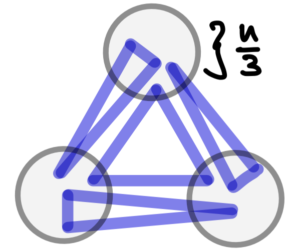

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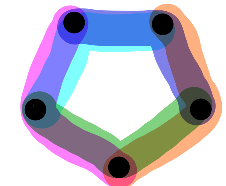

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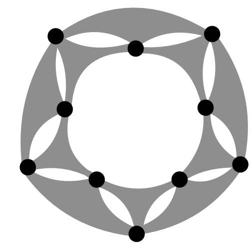

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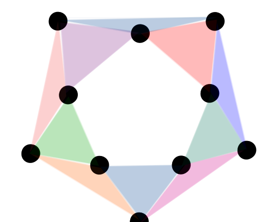

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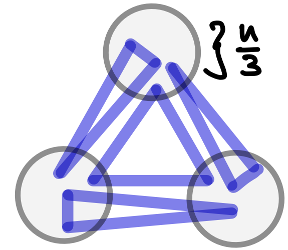

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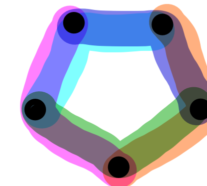

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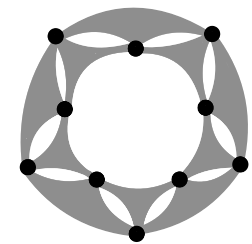

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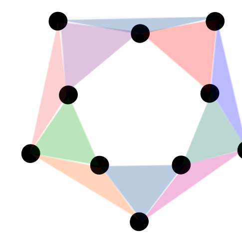

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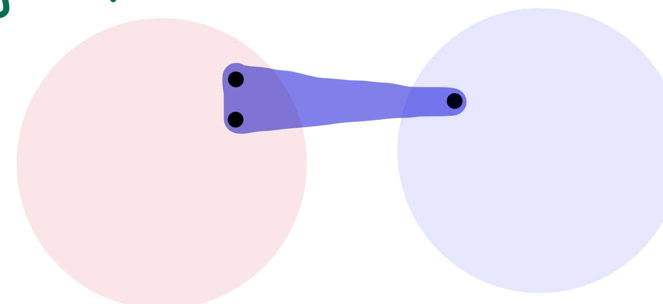

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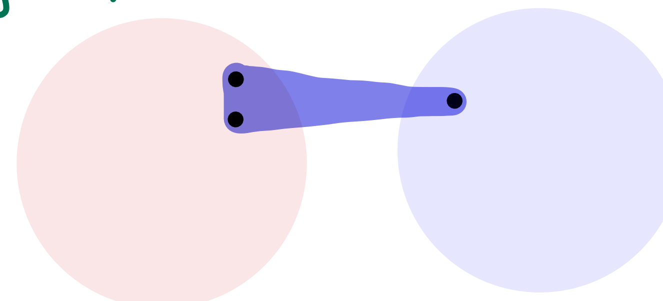

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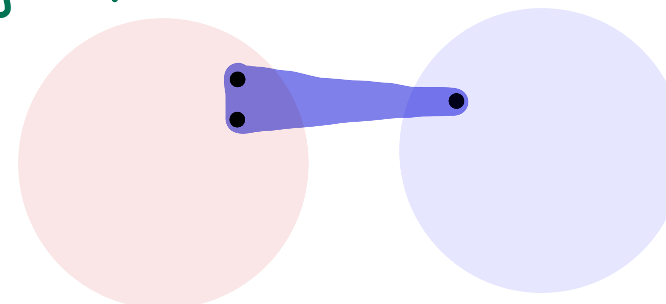

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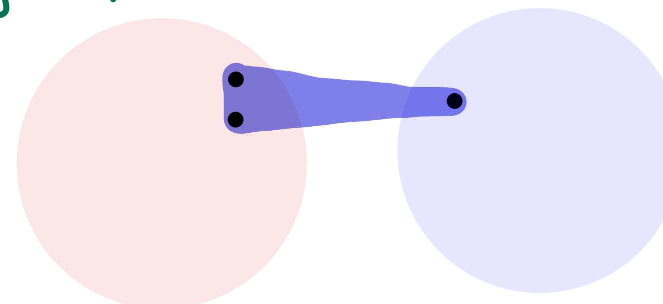

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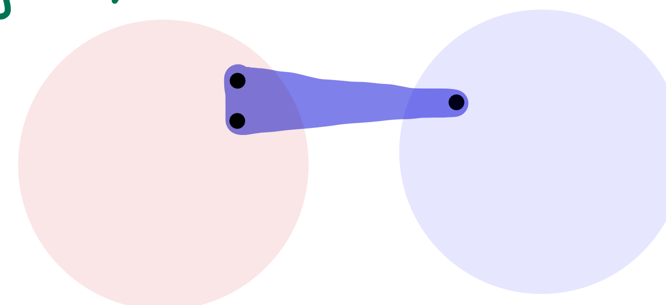

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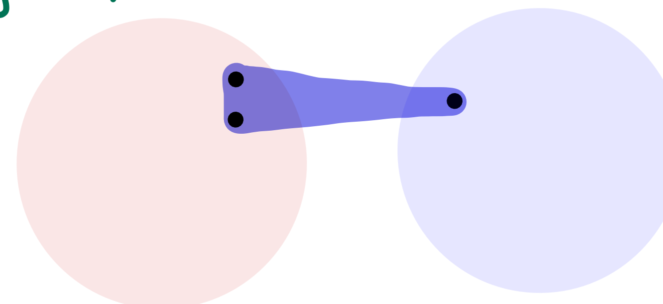

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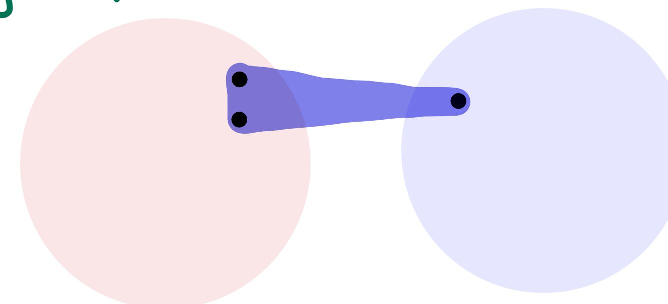

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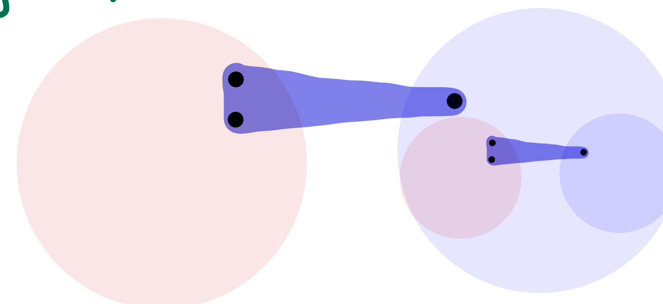

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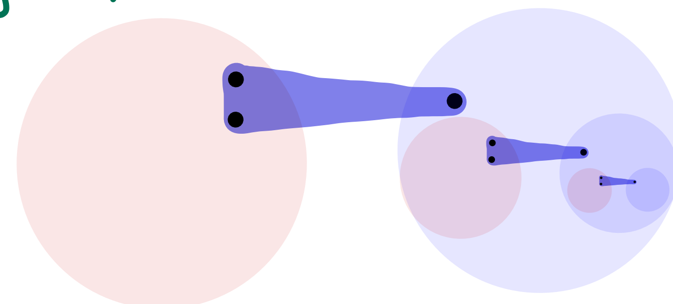

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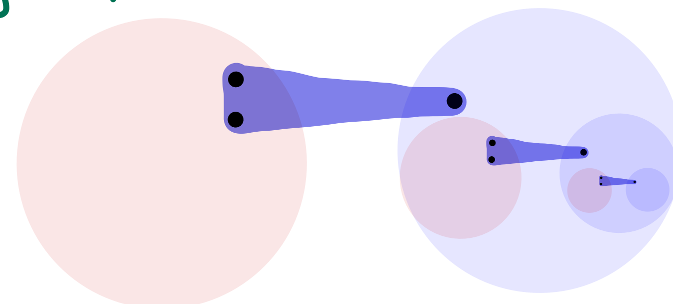

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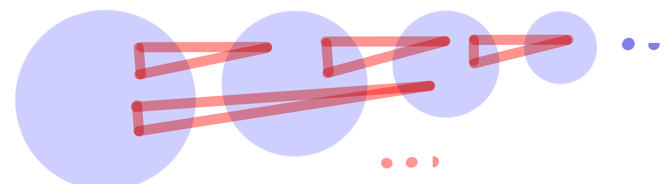

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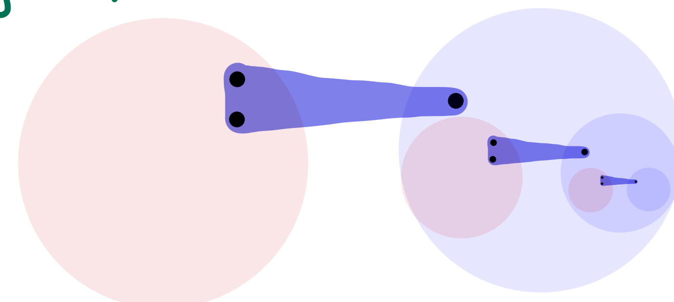

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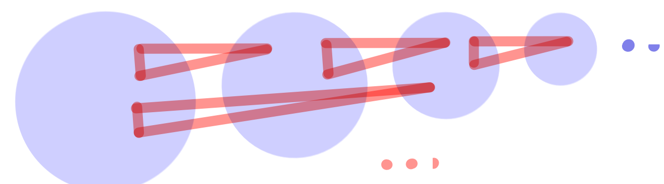

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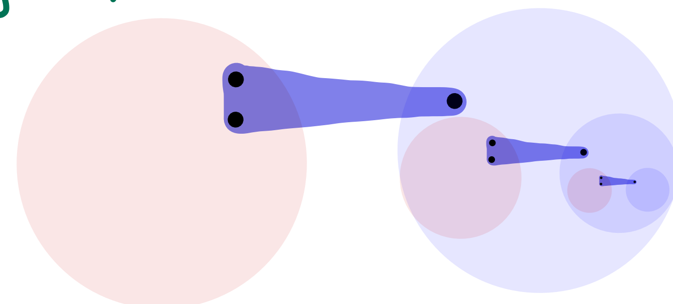

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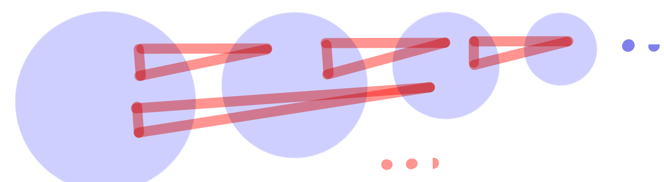

$\Rightarrow \pi(C_\ell^{(3)}) \ge \alpha$ for ℓ odd.

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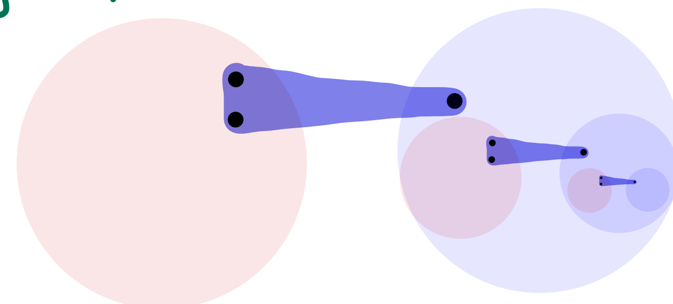

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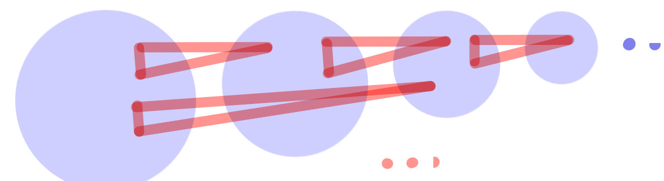

Question: is $\pi(C_\ell^{(3)}) = \alpha$ for odd $\ell \geq 5$?

Our result

Theorem (Kamčev–L.–Pokrovskiy 2022+).

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* One of few known irrational Turán densities (Yan–Peng 2022 and Wu 2022 provide other examples).

Blow-ups and pseudocycles

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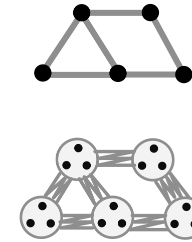

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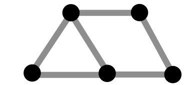

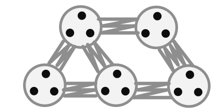

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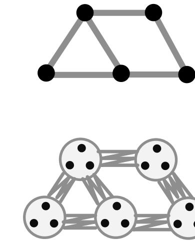

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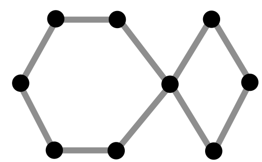

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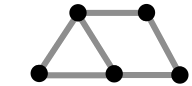
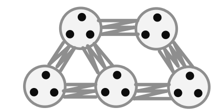
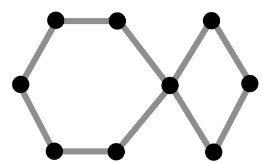




A **pseudocycle** of length ℓ is a cyclic sequence $v_1, \ldots, v_\ell$ s.t. $v_i v_{i+1} v_{i+2}$ is an edge for $i \in \{1, \ldots, \ell\}$ (the v_i's need not be distinct).



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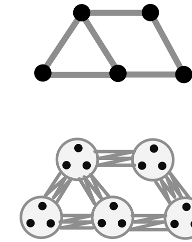

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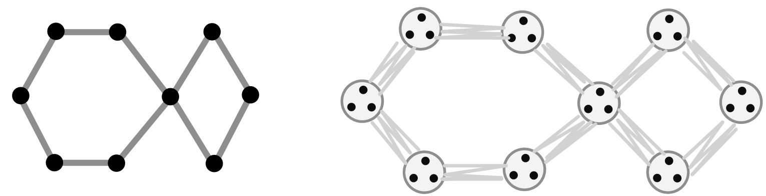

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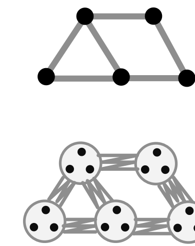

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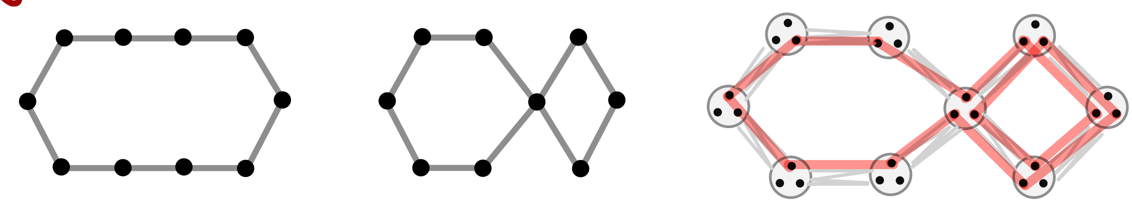

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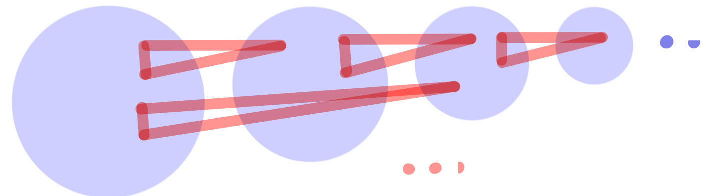

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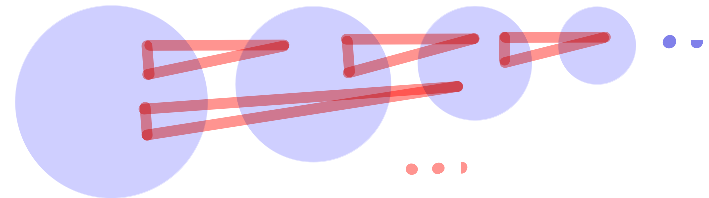

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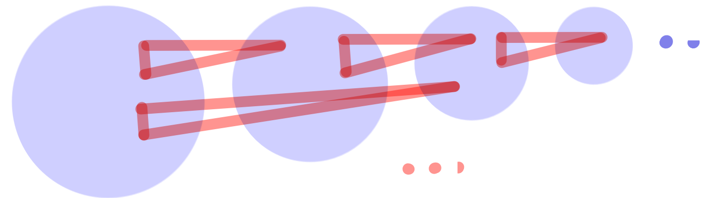

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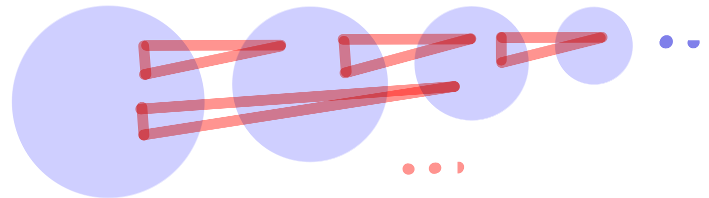

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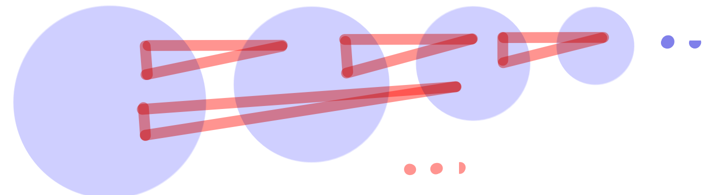

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Theorem (baby version).

Every hypergraph on n vertices with no odd pseudocycles has $\leq f(n)$ edges.

Characterisation of hypergraphs with no odd pseudocycles

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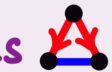
Lemma. A hypergraph has no odd pseudocycles iff its pair of vertices can be coloured blue  and red and oriented  s.t. all edges are cherries .

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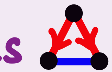
Lemma. A hypergraph has no odd pseudocycles iff its pair of vertices can be coloured blue  and red and oriented  s.t. all edges are cherries .

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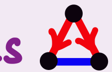
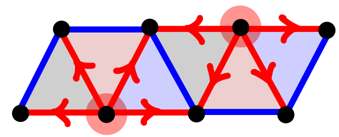
Lemma. A hypergraph has no odd pseudocycles iff its pair of vertices can be coloured blue  and red and oriented  s.t. all edges are cherries .

Call such a colouring good. 

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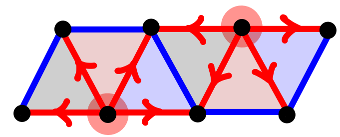

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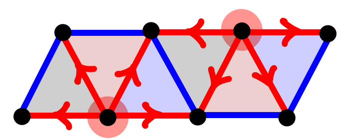

In a good colouring, every third vertex in a pseudopath/pseudocycle is an 'apex', so all pseudocycles have length divisible by 3, proving 'only if' part.

The lemma is a generalisation of: a graph G has no odd cycles iff its vertices can be red-blue coloured s.t. every edge looks like .

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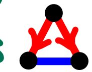
Falgas-Ravry – Vaughan 2012, Huang 2014. Every n-vertex graph whose edges are blue or red and oriented has $\leq f(n)$ cherries  .

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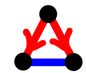
__Falgas-Ravry–Vaughan 2012, Huang 2014.__ Every n-vertex graph whose edges are blue or red and oriented has $\leq f(n)$ cherries  .

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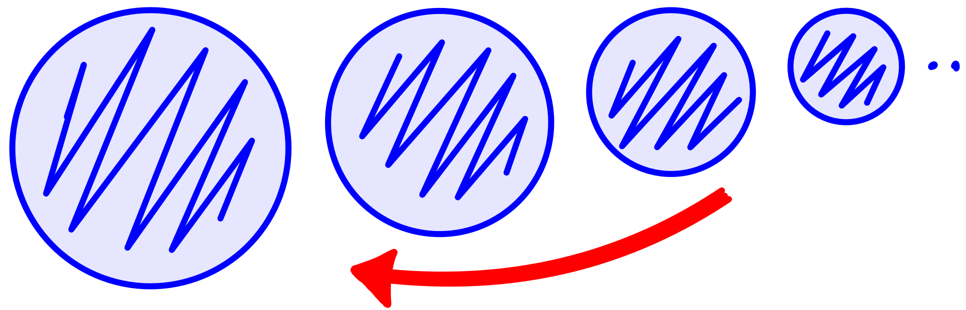

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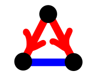
Falgas-Ravry – Vaughan 2012, Huang 2014. Every n-vertex graph whose edges are blue or red and oriented has $\leq f(n)$ cherries .

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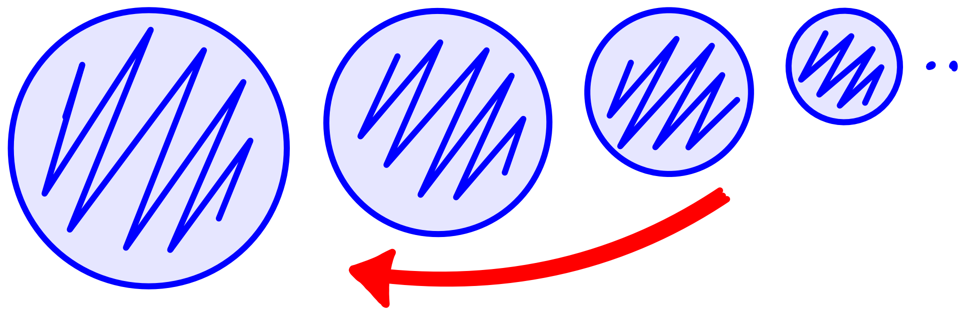

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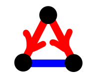
Falgas-Ravry — Vaughan 2012, Huang 2014. Every n-vertex graph whose edges are blue or red and oriented has $\leq f(n)$ cherries  .

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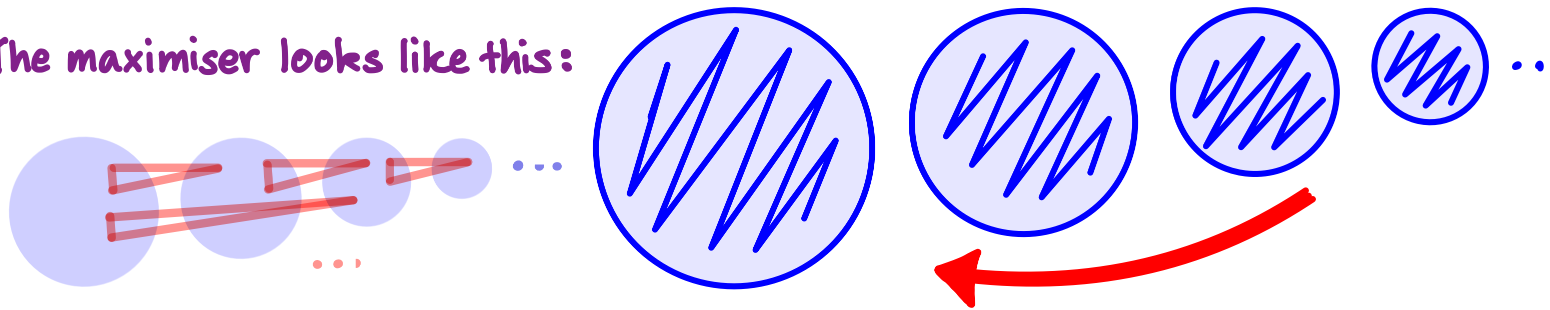

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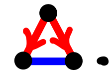
Falgas-Ravry – Vaughan 2012, Huang 2014. Every n-vertex graph whose edges are blue or red and oriented has $\leq f(n)$ cherries  .

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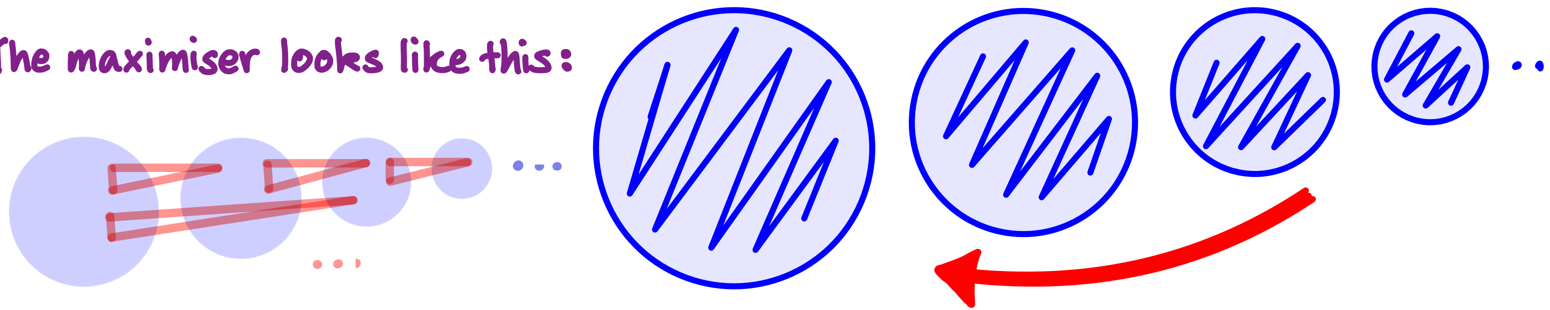

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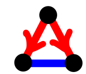
Falgas-Ravry—Vaughan 2012, Huang 2014. Every n-vertex graph whose edges are blue or red and oriented has $\leq f(n)$ cherries .

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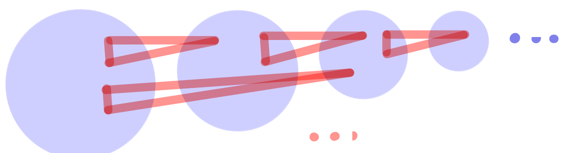

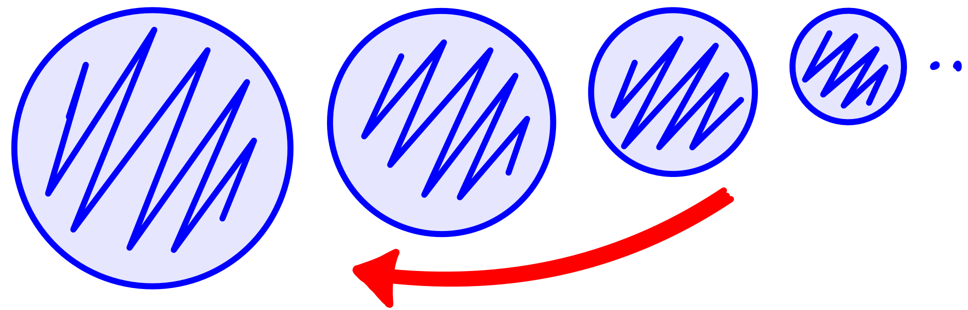

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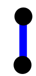
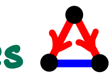
  $\Rightarrow$ pairs of vertices can be coloured  and  s.t. all edges are cherries  .

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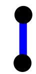
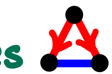
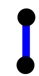
  $\Rightarrow$ pairs of vertices can be coloured  and  s.t. all edges are cherries .

* An n-vertex graph whose edges are coloured  and  has $\leq f(n)$ cherries.

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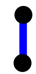
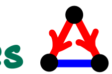
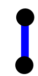
  $\Rightarrow$ pairs of vertices can be coloured  and  s.t. all edges are cherries .

* An n-vertex graph whose edges are coloured  and  has $\leq f(n)$ cherries.

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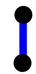
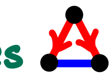
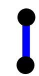
  $\Rightarrow$ pairs of vertices can be coloured  and  s.t. all edges are cherries .

* An n-vertex graph whose edges are coloured  and  has $\leq f(n)$ cherries.

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Want:

Stability

Stability

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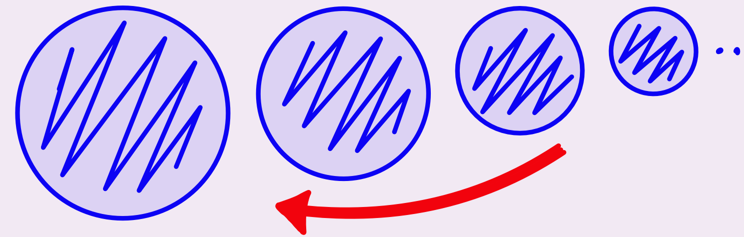

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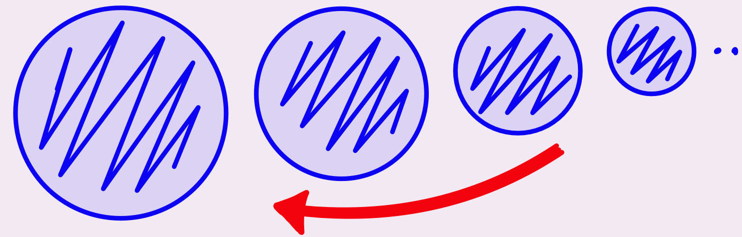
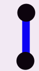
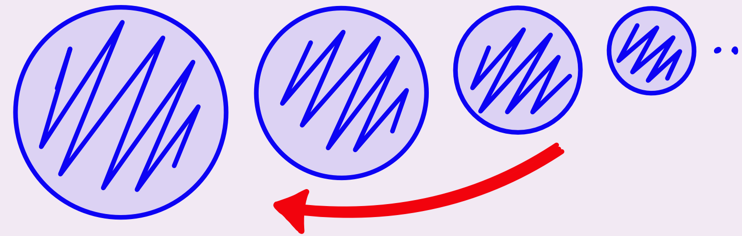


Liu-Pikhurko-Sharifzadeh-Staden (2020) provide a scheme for proving stability versions of extremal results proved by symmetrisation.

Stability

Lemma. Let G be an n-vertex graph with edges coloured  and which has $\geq f(n) - \varepsilon n^3$ cherries. Then G is close to:



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[This does not apply here because our extremal example has a varying number of parts.]

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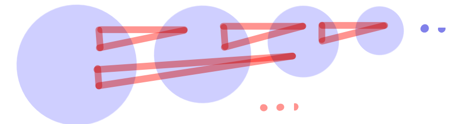

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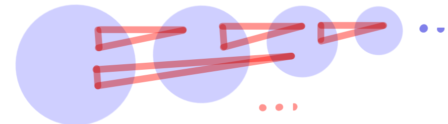

Aim: show that $\mathcal{H}$ has $\leq f(n) + O(1)$ edges.

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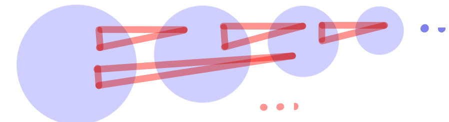

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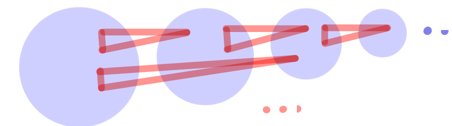

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* $\Rightarrow$ (Characterisation lemma) There is a graph G on $V(\mathcal{H}')$, whose edges are coloured and, s.t. edges in $\mathcal{H}'$ are cherries in G.

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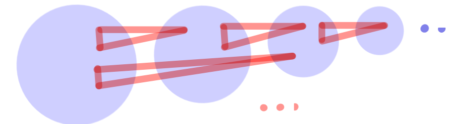

Aim: show that $\mathcal{H}$ has $\leq f(n) + O(1)$ edges.

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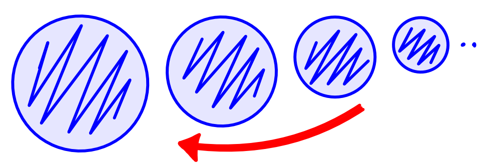
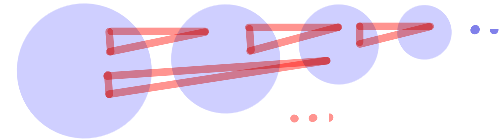


Proof of main step - continued

$* \Rightarrow$ $\mathcal{H}'$, and thus $\mathcal{H}$, are close to:



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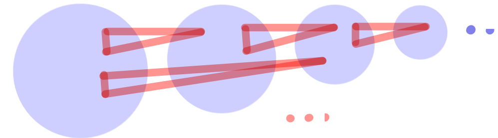
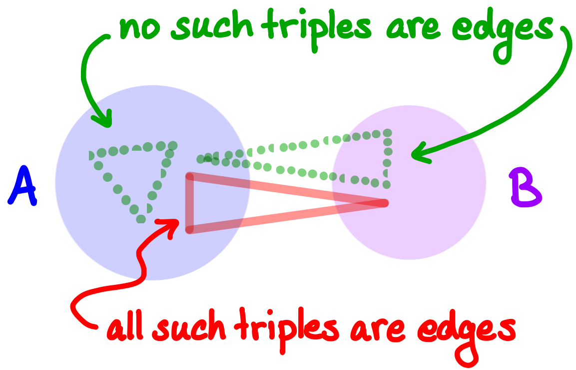
* $\Rrightarrow$ $\mathcal{H}'$, and thus $\mathcal{H}$, are close to: 

* $\Rrightarrow$ (Difficult) there is a partition $\{A,B\}$ of $V(\mathcal{H})$, with $A,B \neq \emptyset$, s.t.:



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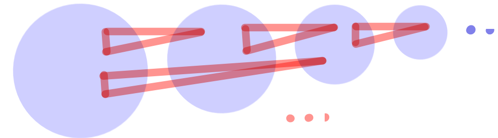

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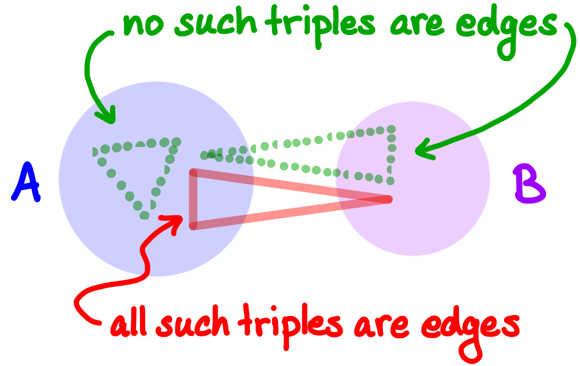

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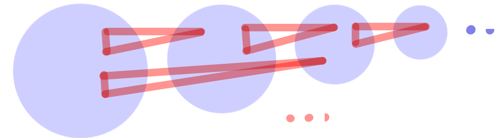

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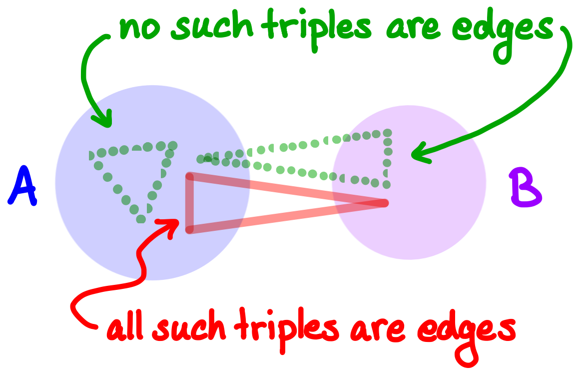

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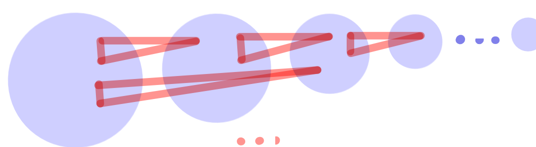

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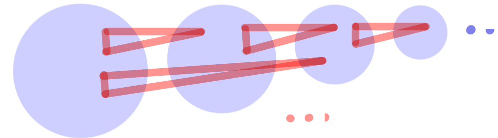
Proof of main step – continued

* ⇒ $\mathcal{H}'$, and thus $\mathcal{H}$, are close to: 
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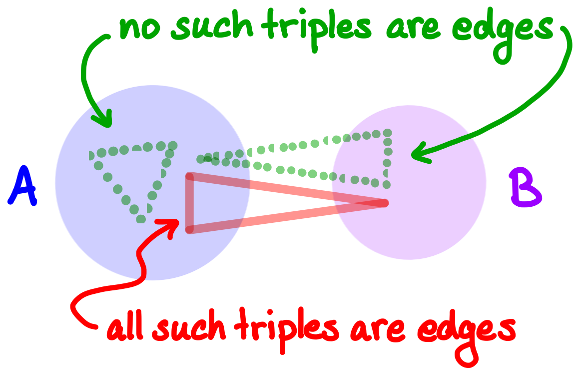

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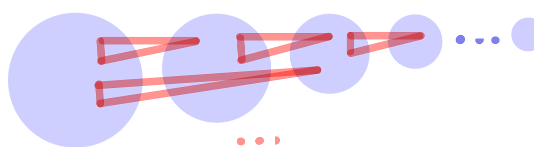
* ⇒ $\mathcal{H}$ looks like 
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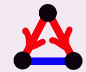
Lemma. A hypergraph has no odd pseudocycles iff its pair of vertices can be coloured  and  s.t. all edges are cherries .

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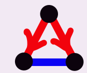
Lemma. A hypergraph has no odd pseudocycles iff its pair of vertices can be coloured  and  s.t. all edges are cherries .

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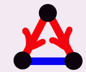
Lemma. A hypergraph has no odd pseudocycles iff its pair of vertices can be coloured  and  s.t. all edges are cherries .

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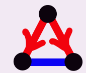
Lemma. A hypergraph has no odd pseudocycles iff its pair of vertices can be coloured  and  s.t. all edges are cherries .

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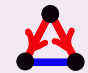
Lemma. A hypergraph has no odd pseudocycles iff its pair of vertices can be coloured  and  s.t. all edges are cherries .

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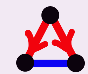
Lemma. A hypergraph has no odd pseudocycles iff its pair of vertices can be coloured  and  s.t. all edges are cherries .

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Thank you for listening!