Twin-width and its implications

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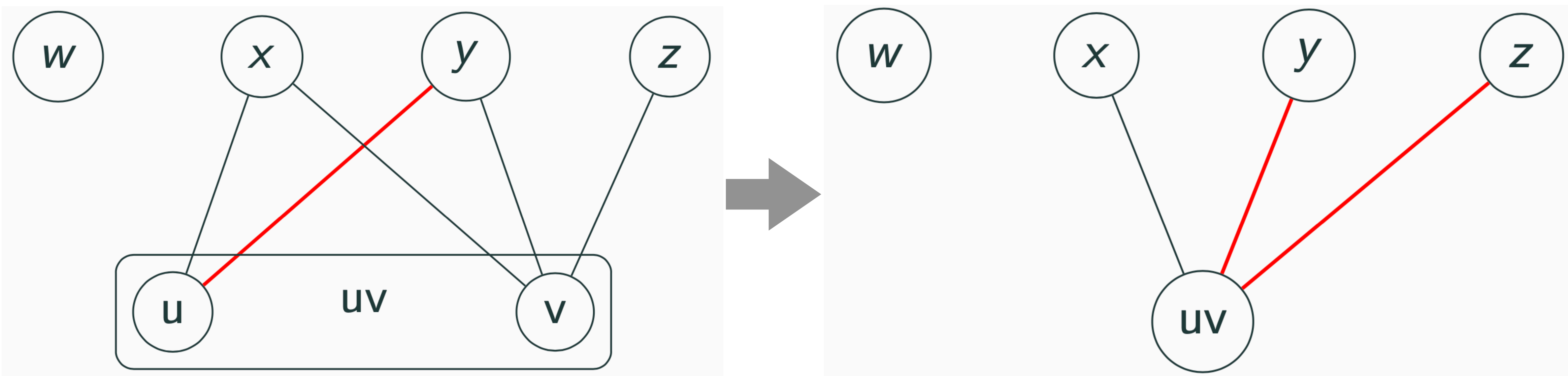
Contraction in a trigraph

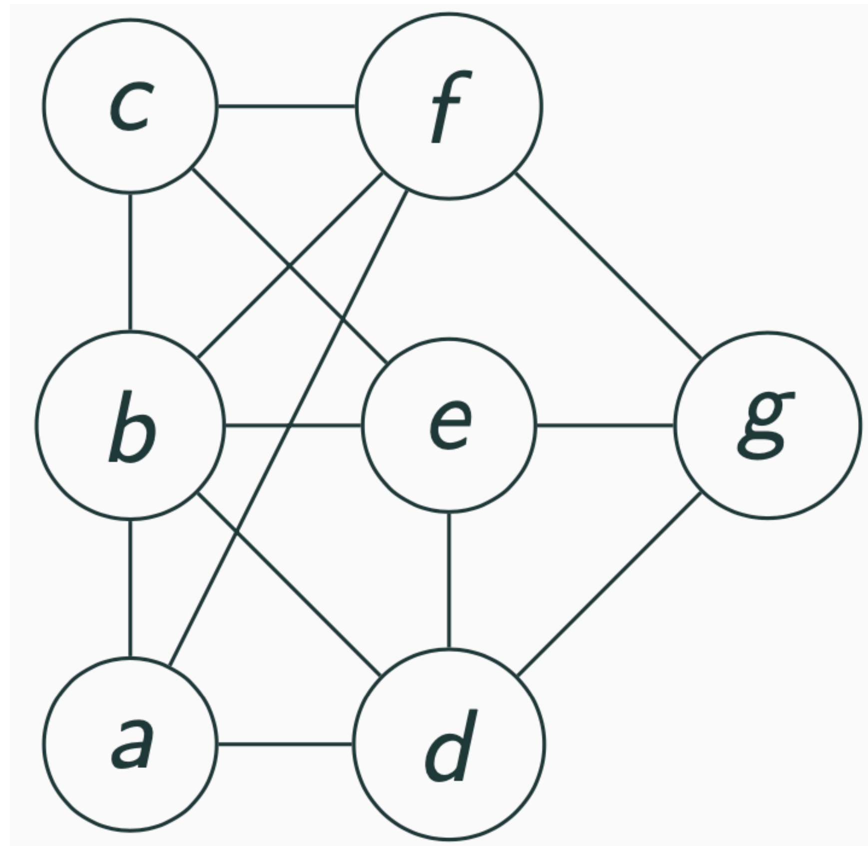
Trigraph has three types of adjacency: (black) edge, non-edge, red edge

Identification of two vertices, not-necessarily adjacent



- edges with $N(u) \triangle N(v)$ turn red
- red edges stay red

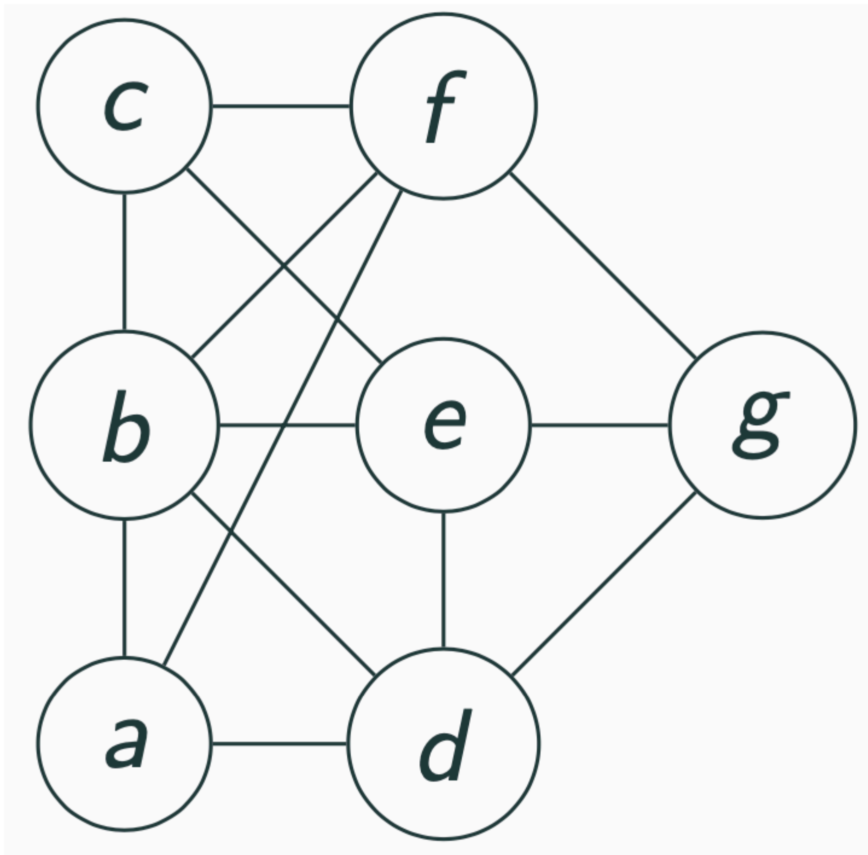
Contraction Sequence



A contraction sequence of G =

a sequence of trigraphs $G = G_n, G_{n-1}, \ldots, G_1$= single-vertex graph
such that G_i is obtained from G_{i+1} by one contraction

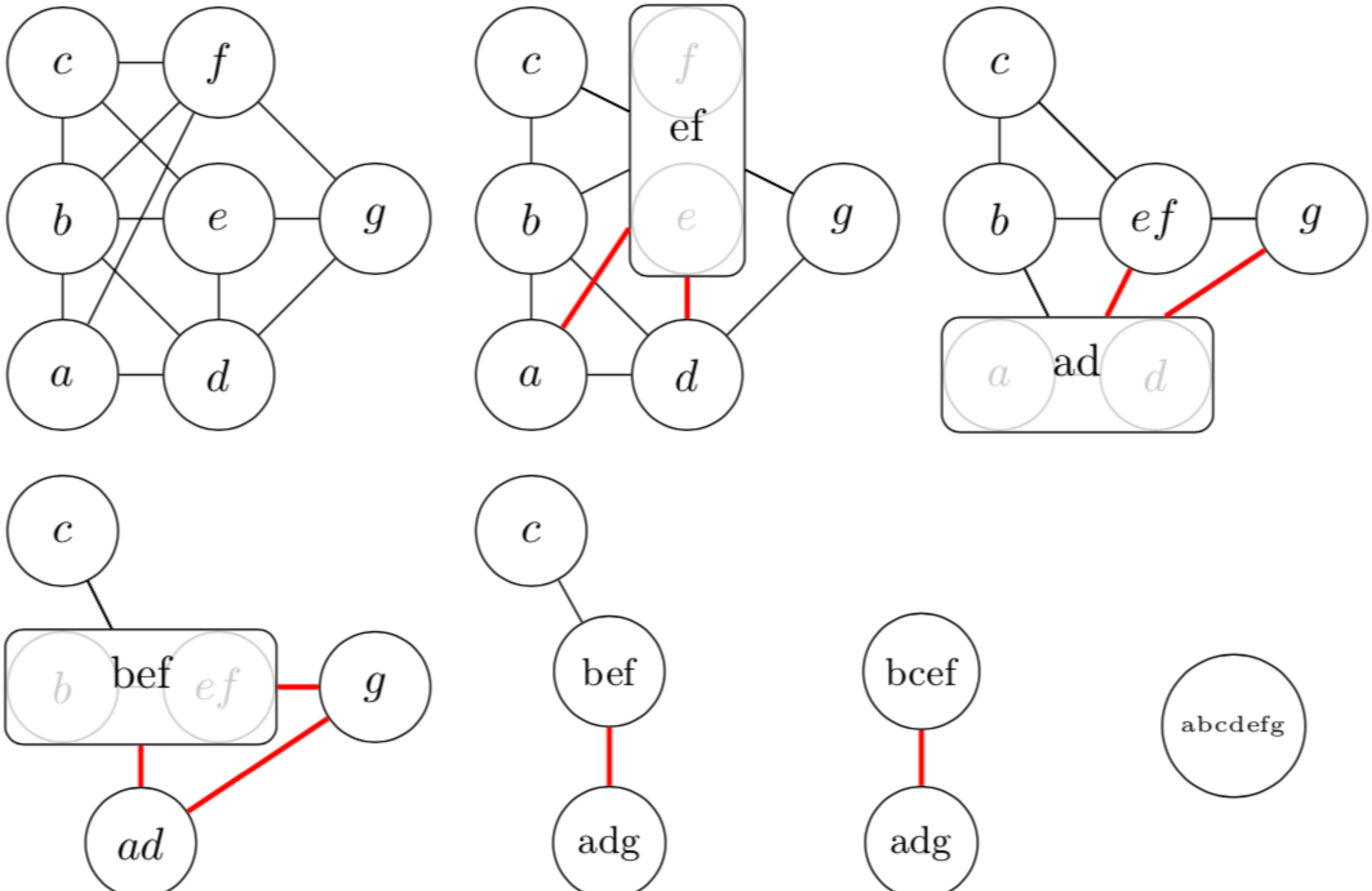
Contraction Sequence



A d-contraction sequence of G =

a sequence of trigraphs $G = G_n, G_{n-1}, \ldots, G_1$= single-vertex graph
such that G_i is obtained from G_{i+1} by one contraction
and the max red degree of each G_i is at most d.

2-contraction sequence



Twin-width of a graph

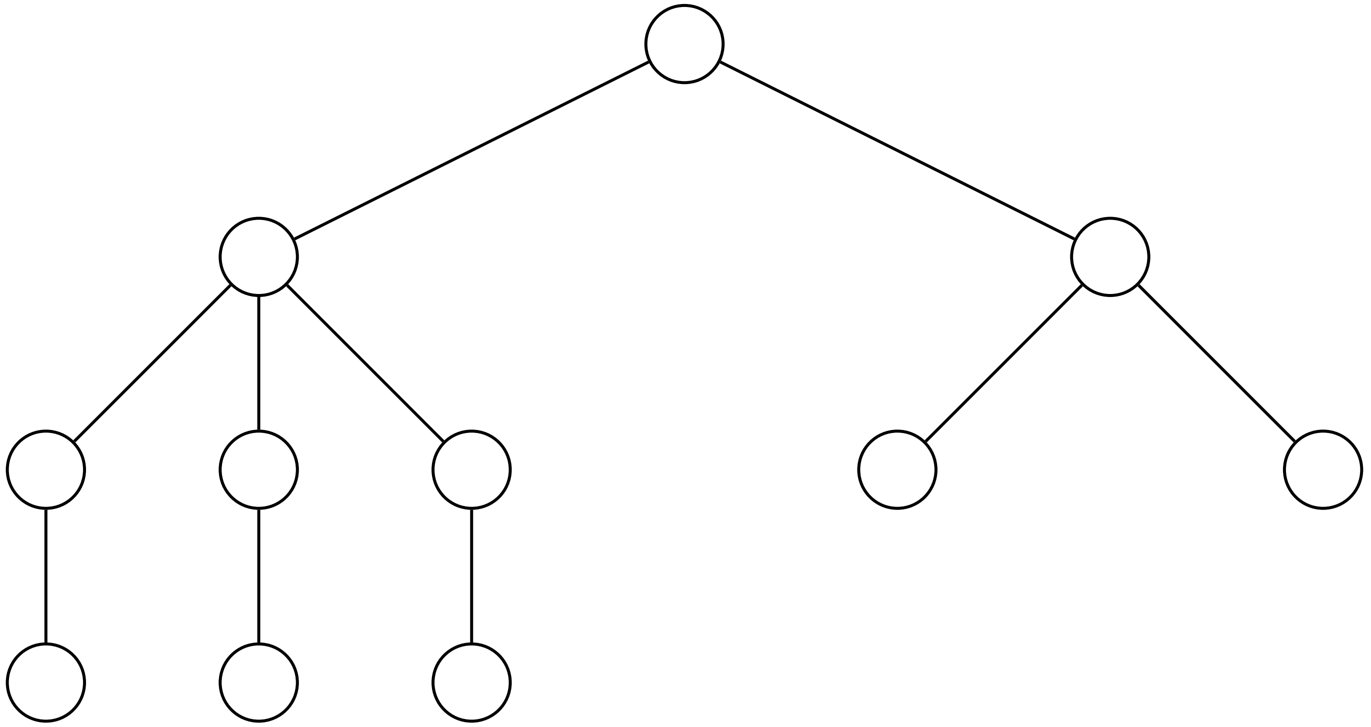
Twin-width of G =

the smallest d s.t. ∃ d-contraction sequence of G.

What is the (upper-bound of) twin-width of …

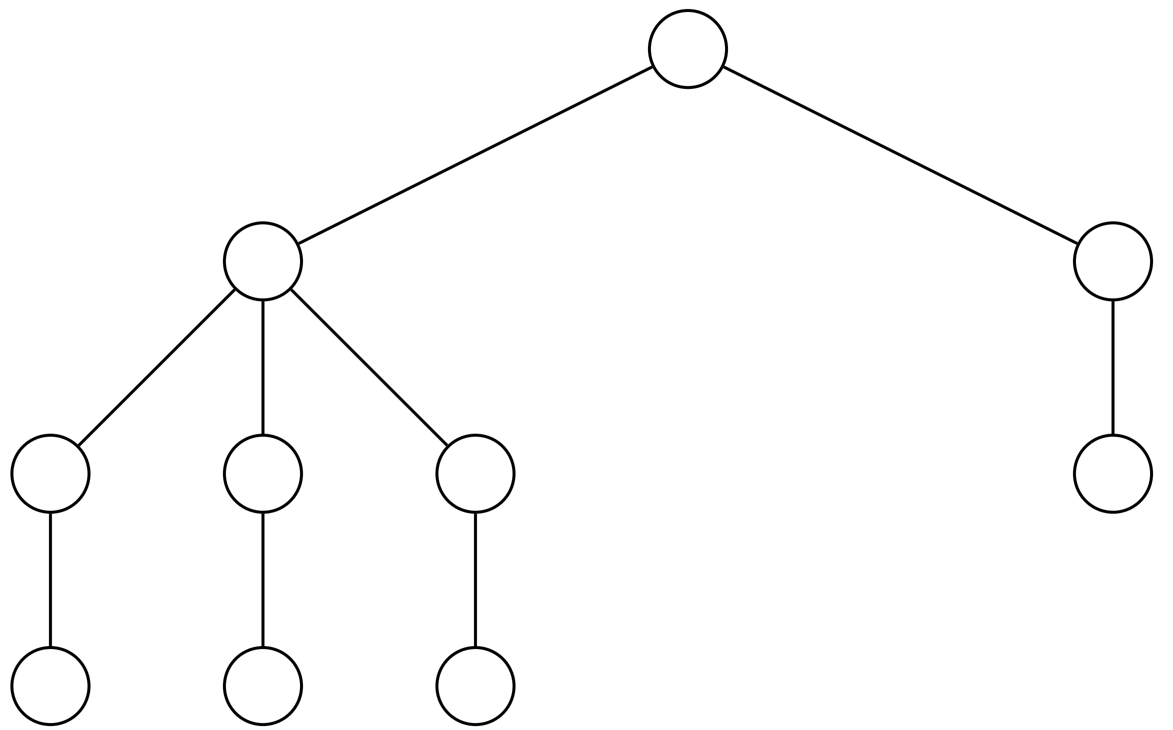
- clique?
- disjoint union of G and H?
- complete join of G and H?
- cograph?
- path?
- tree?

Trees



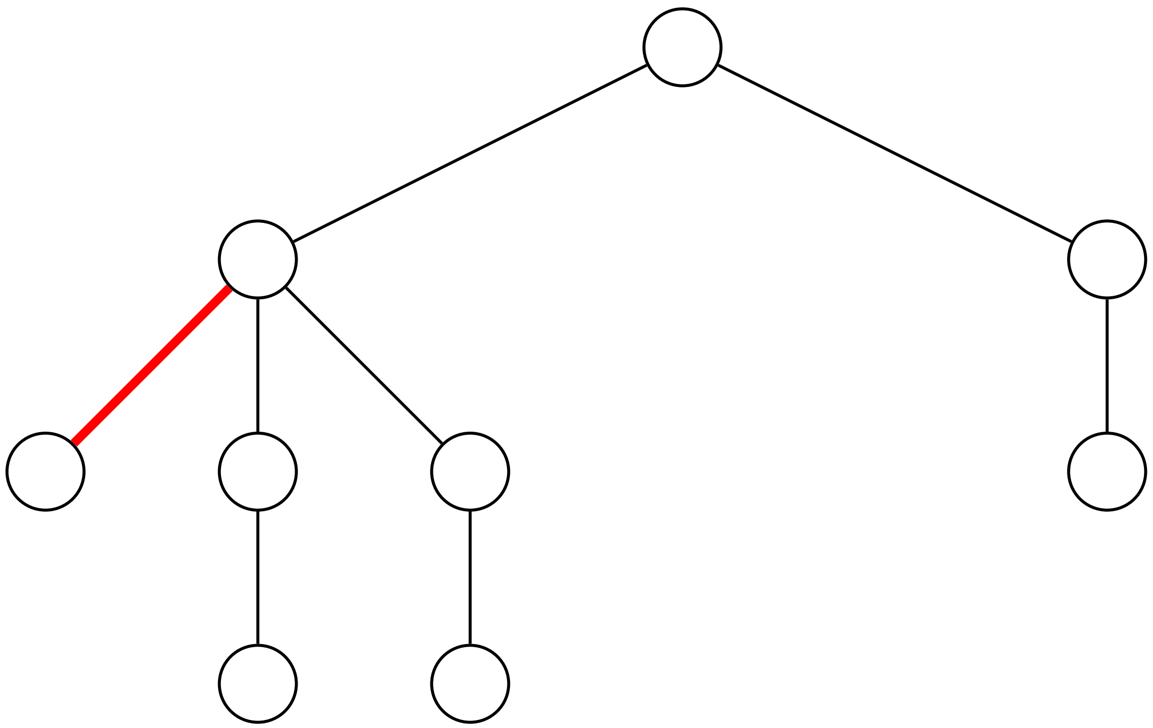
If possible, contract two twin leaves

Trees



If not, contract a deepest leaf with its parent

Trees



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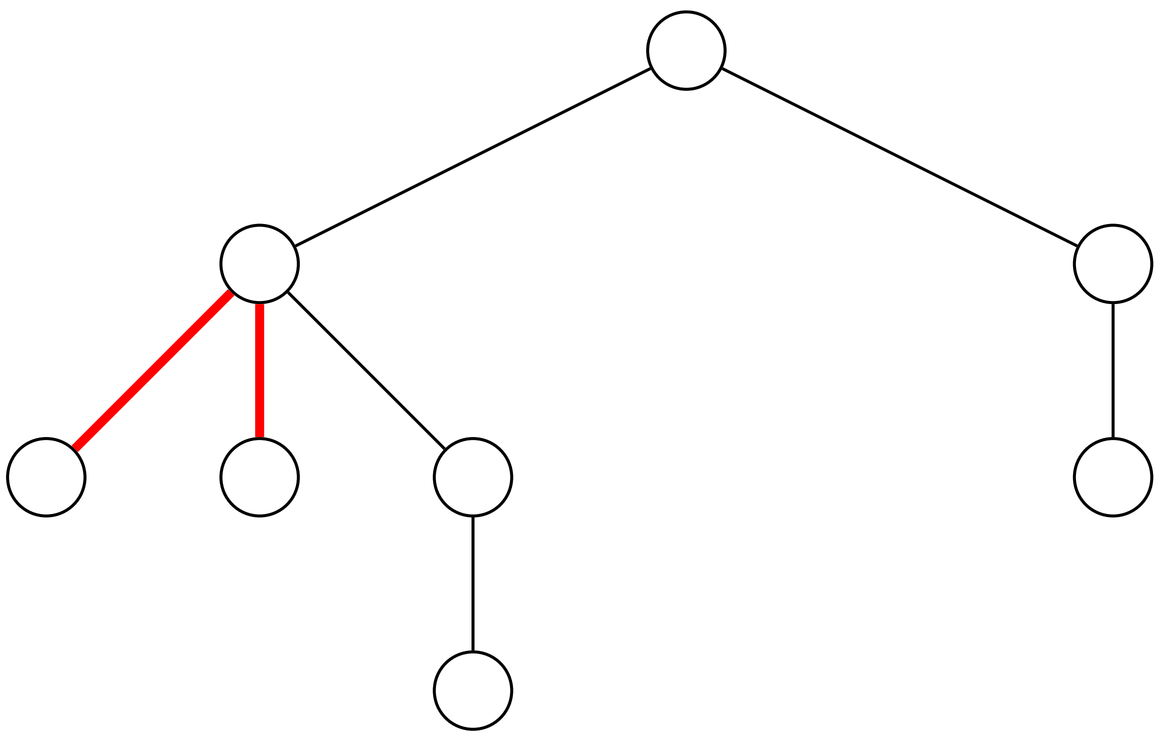

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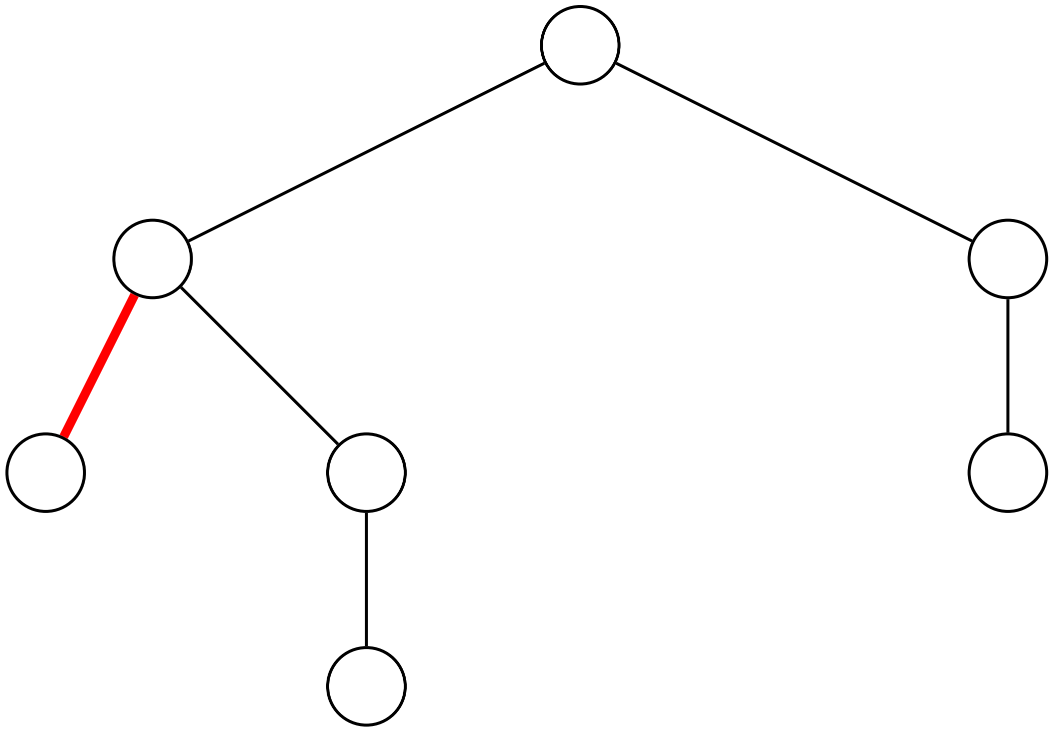
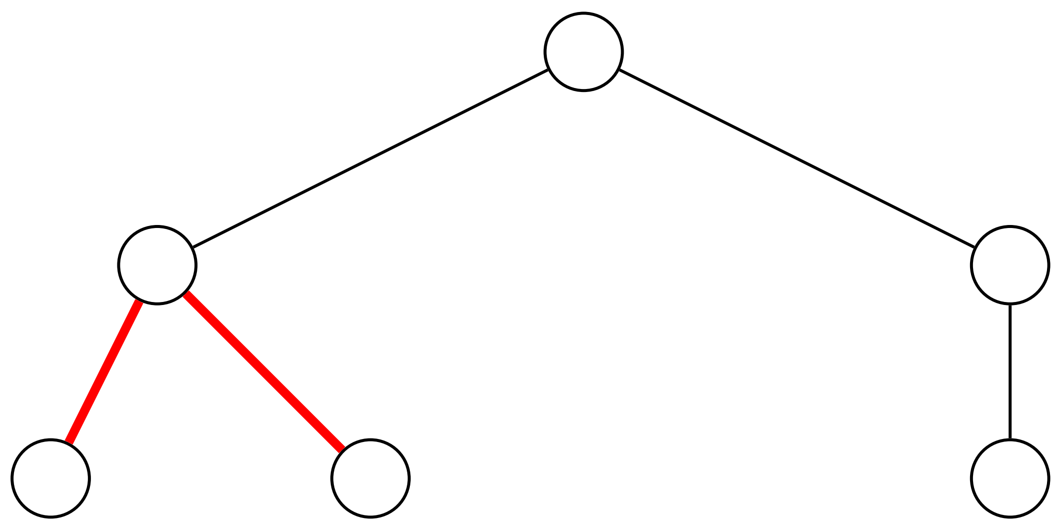


Cannot create a red degree-3 vertex

Trees



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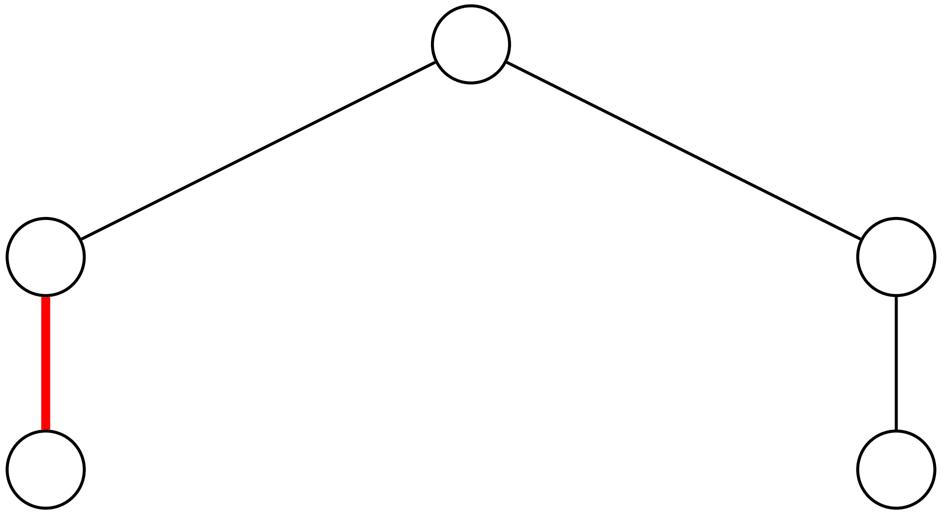

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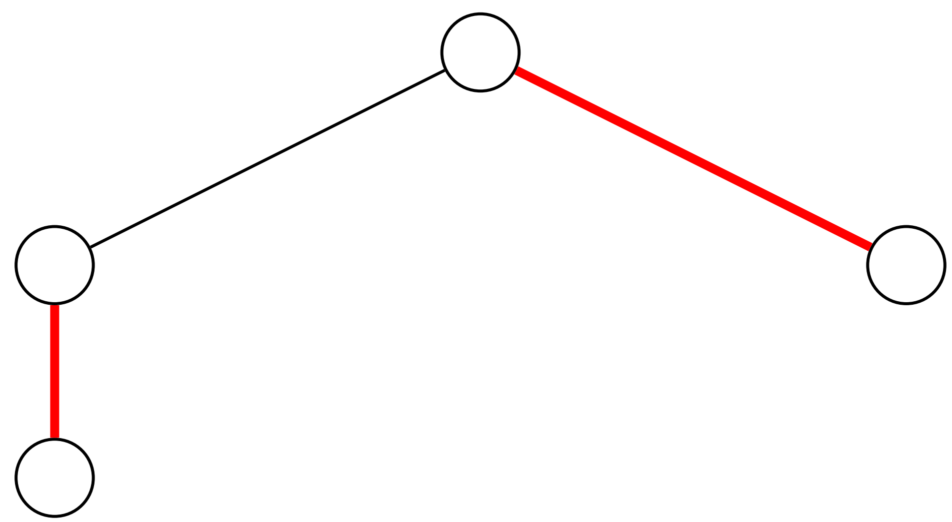

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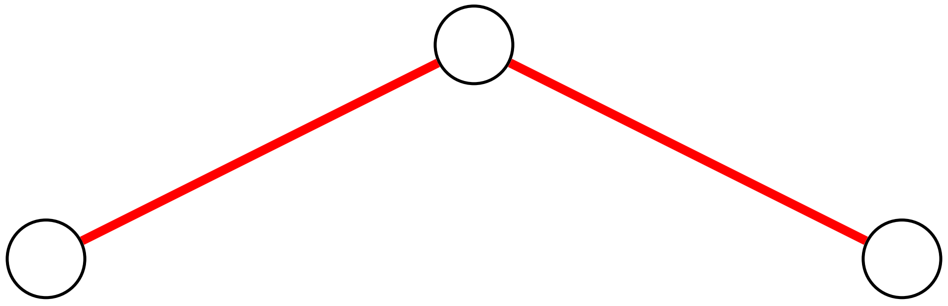

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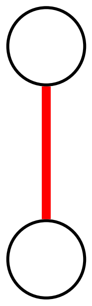

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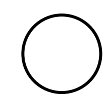
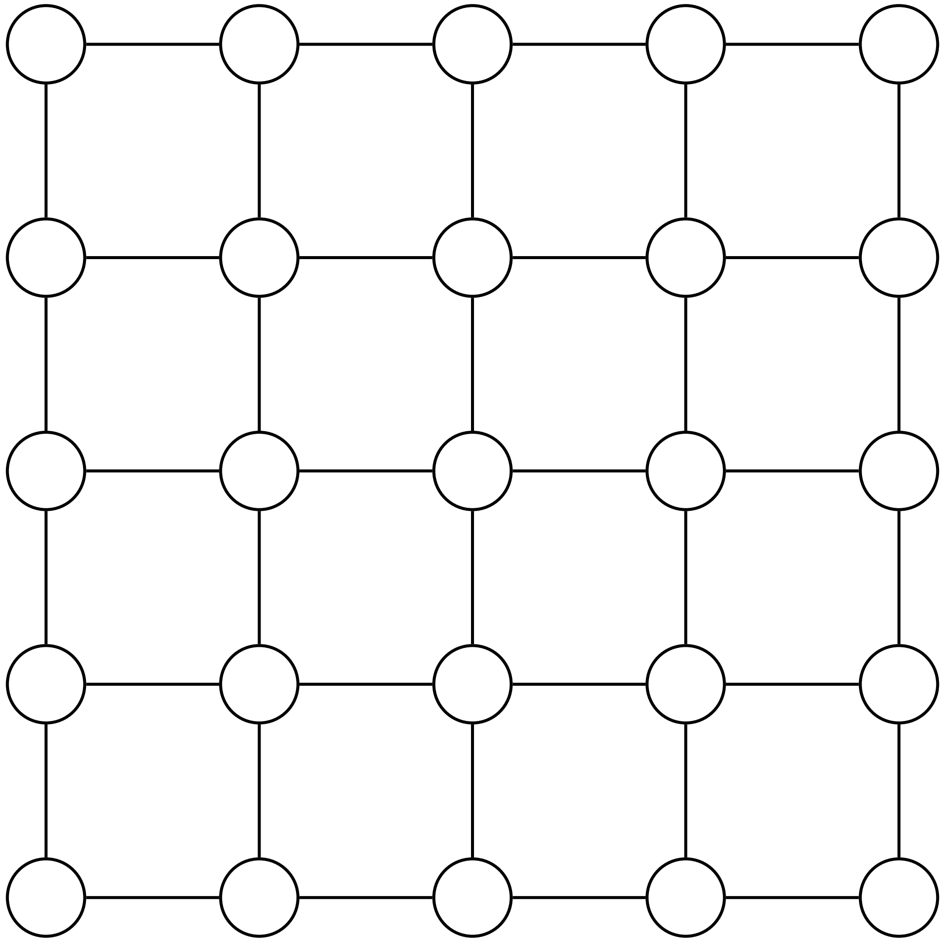


Generalization to bounded *treewidth* and even bounded *rank-width*

Grids



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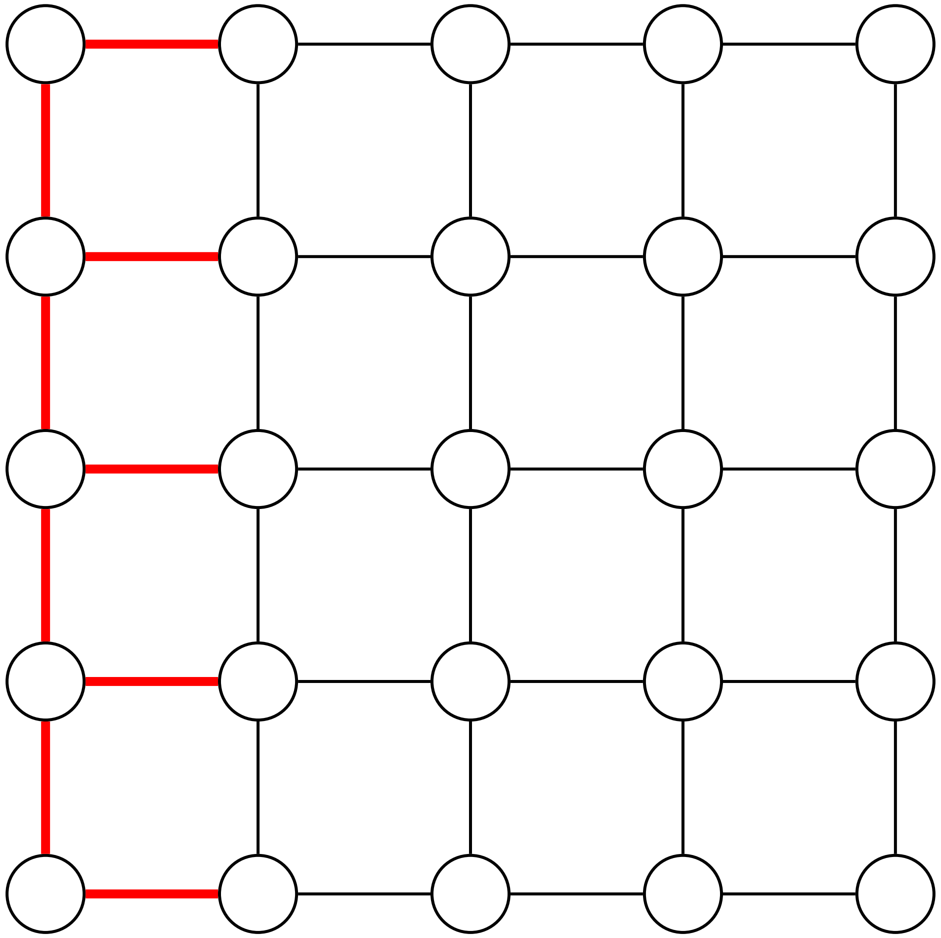

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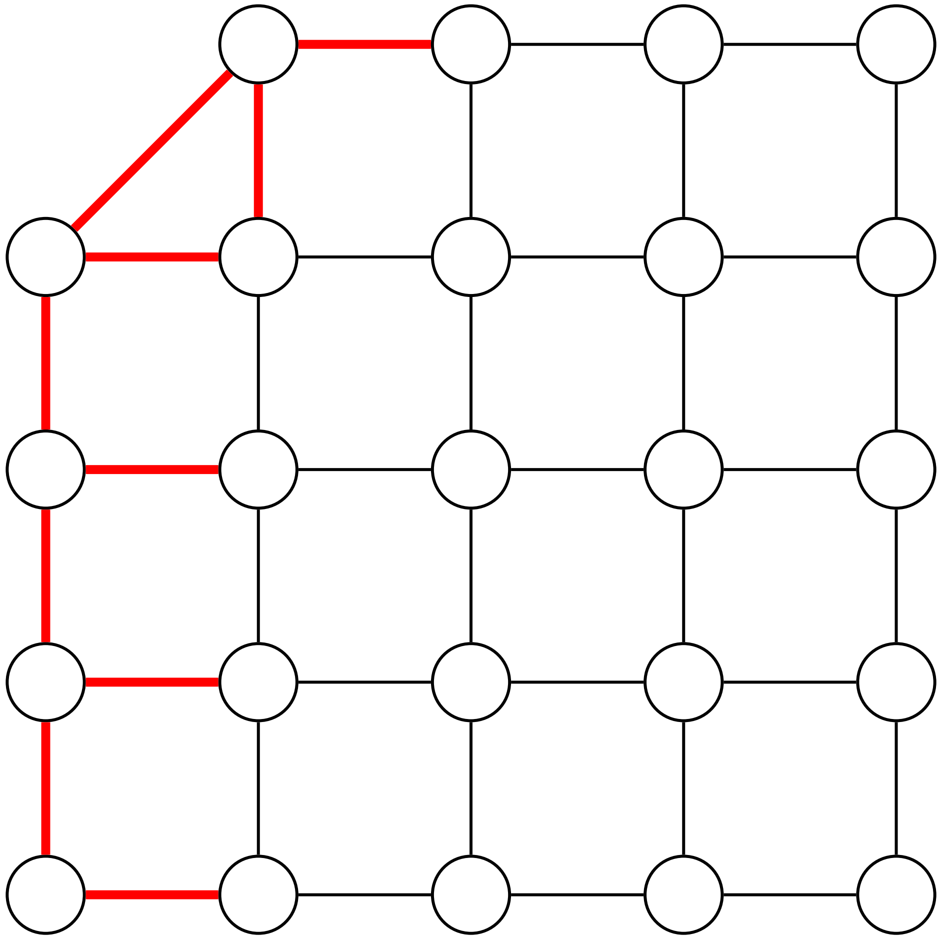

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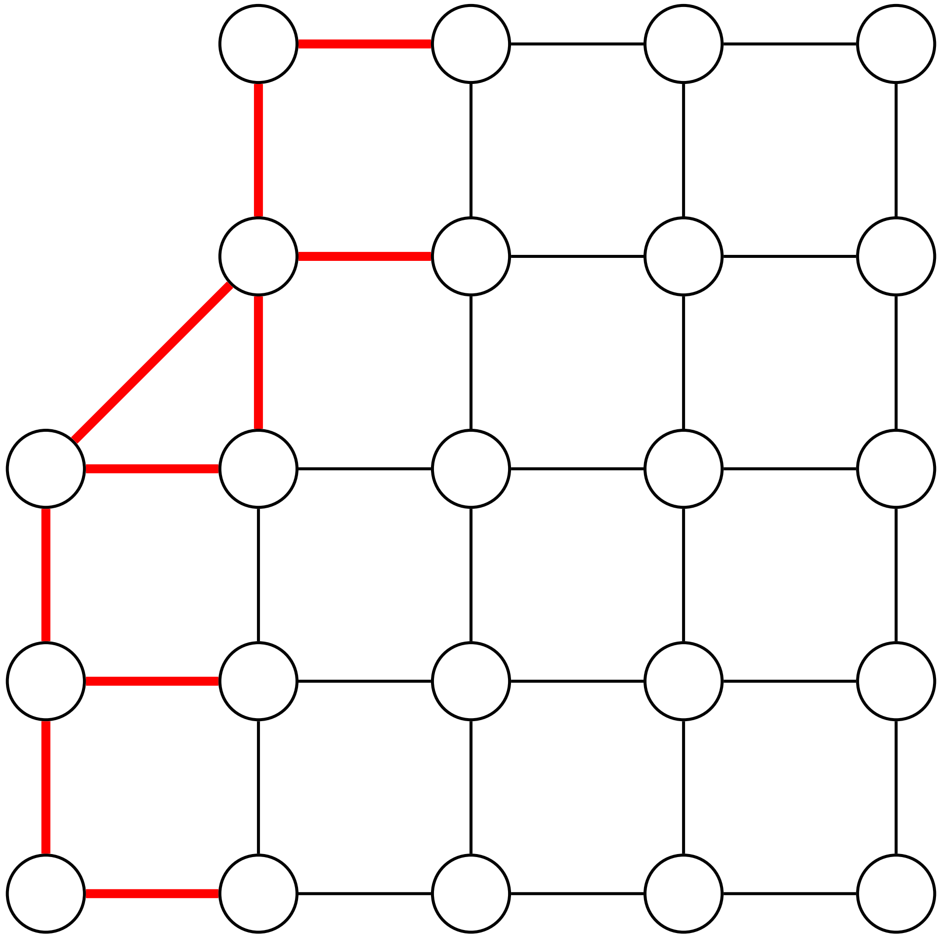

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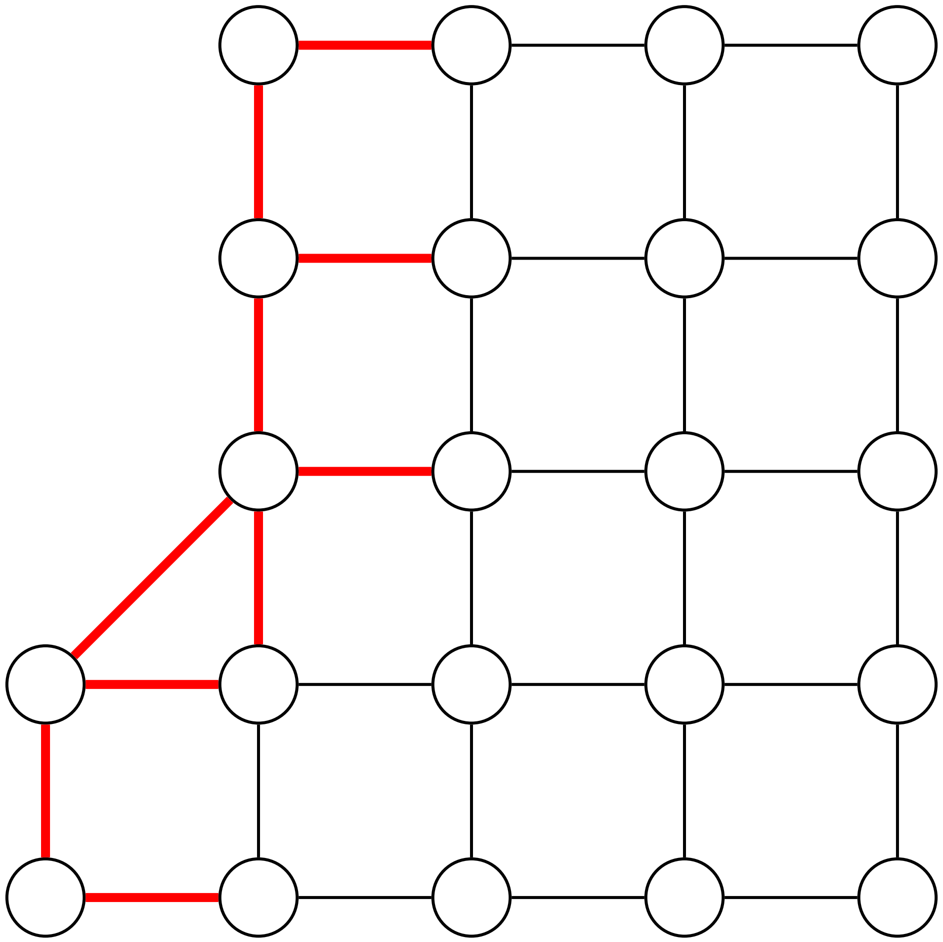

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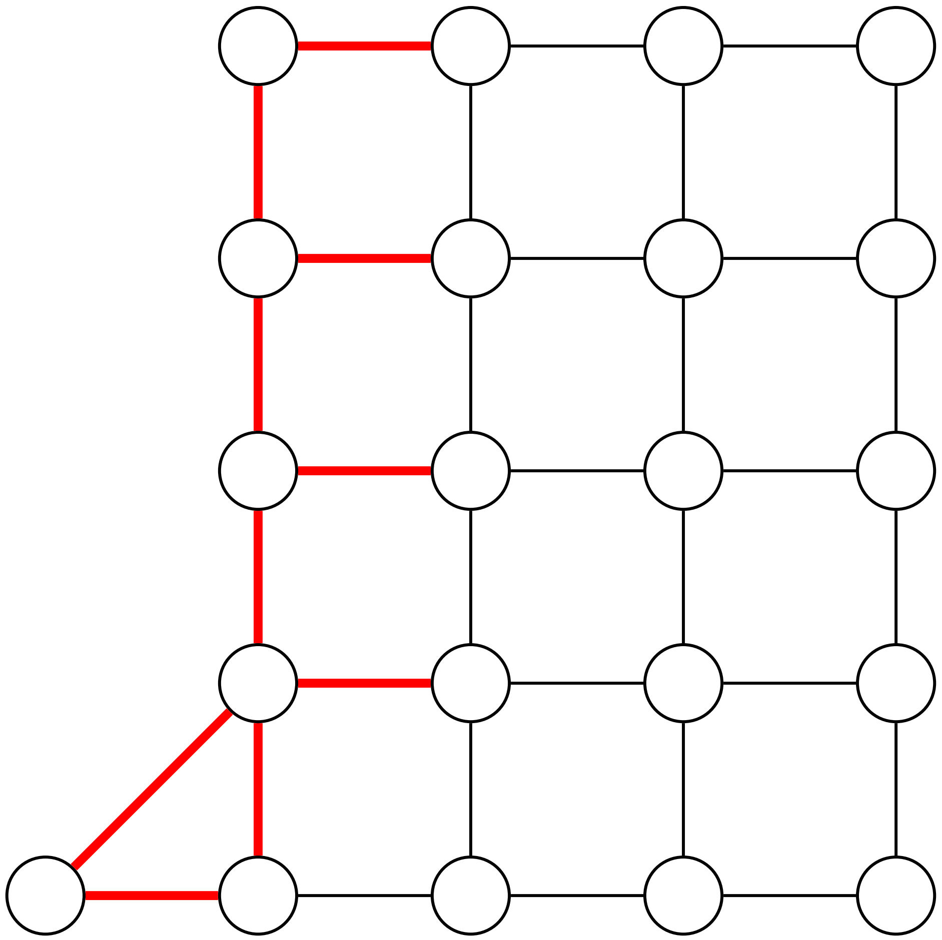

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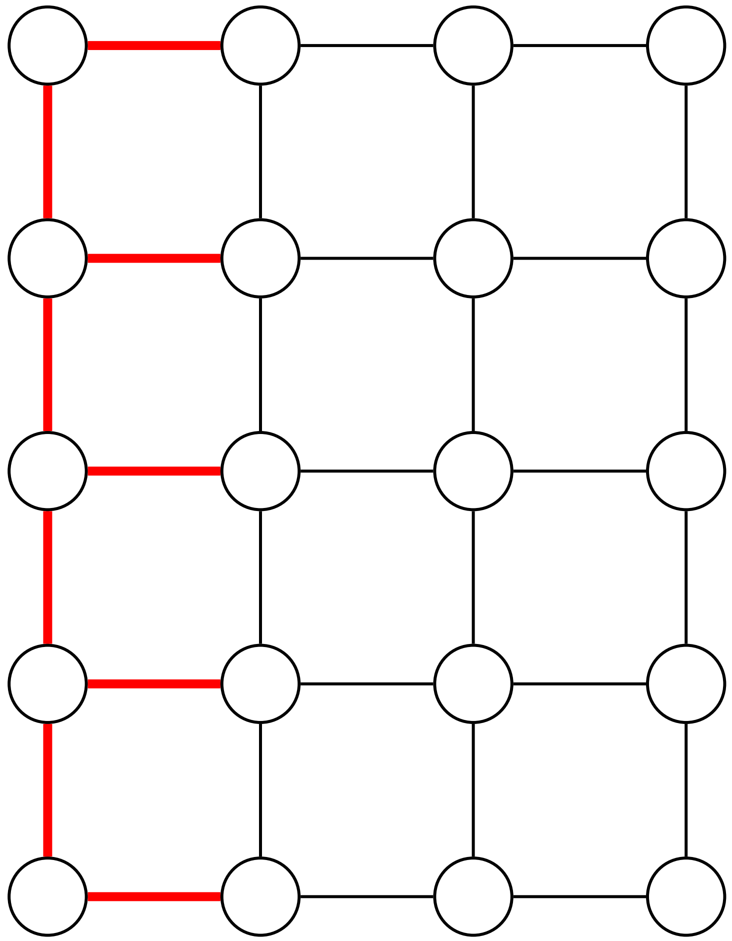

4-sequence for planar grids

Key Messages

1. Twin-width captures many known graph classes, both spare and dense.

2. With twin-width, there is a rich toolbox to investigate graph properties, be it algorithmic, structural, or logical.

3. There are much to be done (by you).

Graph classes of small twin-width

[Bonnet, Geniat, K, Thomassé, Watrigant ’20, ’21]

- trees, graphs of bounded tree-width
- bounded clique-width (rank-width) graphs
- unit interval graphs
- strong products of two graphs of bounded tww, one with bounded degree
- $\Omega(\log n)$-subdivision of all n-vertex graphs, etc.
- (subgraphs of) d-dimensional grids
- K_t-free unit ball graphs in dimension d
- hereditary proper subclass of permutation graphs
- posets of bounded antichain size
- K_t-minor-free graphs
- square of planar graphs
- map graphs
- k-planar graphs
- bounded degree string graphs

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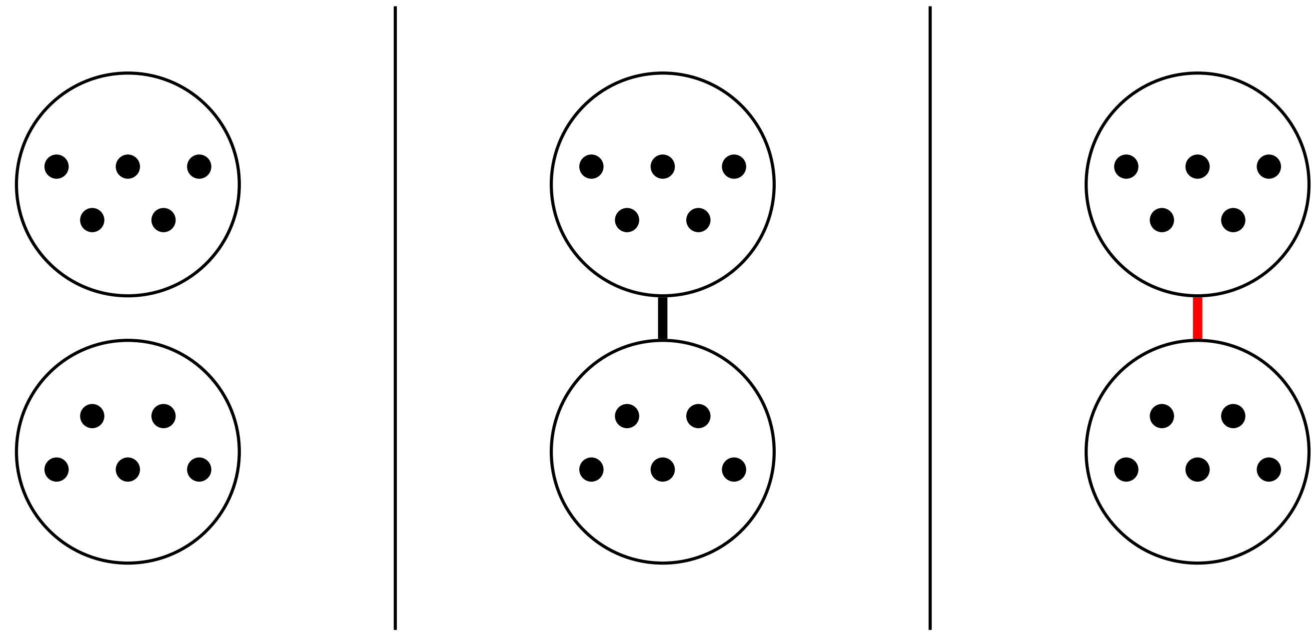
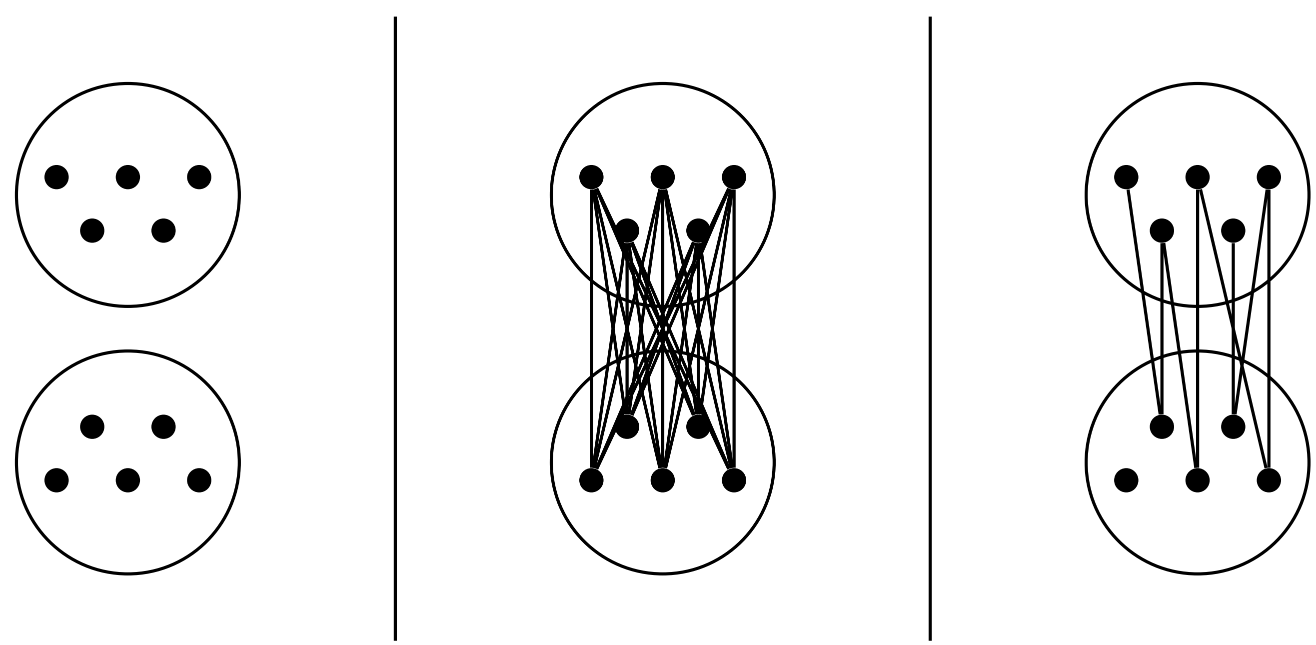
The class of all cubic graphs have unbounded twin-width given two bags:



it means in the original graph:



no edge | all edges | at least one edge, at least one non-edge

Twin-width of a graph

A d-contraction sequence of G =

a sequence of partitions
$\mathscr{P}_n = \{\{v\} : v \in V(G)\}, \mathscr{P}_{n-1}, \ldots, \mathscr{P}_i, \ldots, \mathscr{P}_1 = \{V(G)\}$ such that $\mathscr{P}_i$ is obtained from P_{i+1} by merging two parts
and the max red degree of each quotient graph $G/\mathscr{P}_i$ is at most d.

Twin-width of G =

the smallest d s.t. ∃ d-partition sequence of G.

Stable under basic operations

- Closed under complement: $tww(G) = tww(\bar{G})$
- $tww(H) \leq tww(G)$ if H is an induced subgraph of G
- Color an arbitrary vertex set $U \subseteq V(G)$ and add an apex to U.
  $tww(G^U) \leq 2 \cdot tww(G)$

- $tww(G \boxtimes H) \leq f(tww(G), tww(H), \Delta(H))$
- Taking a subgraph can increase the twin-width arbitrarily.
- If G is $K_{t,t}$-free for some t: $tww(G') \leq f(tww(G), t)$ for $G' \subseteq G$

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Product Structure Theorem for graphs of Euler genus g
[Dujmovič, Joret, Micek, Morin, Ueckerdt, Wood 2020]

Every graph of Euler genus g is a subgraph of

$$H \boxtimes P \boxtimes K_{\max\{2g,3\}}$$

where H is an apex graph of tree-width at most 4, P a path.

Bounds for graphs on surfaces

Planar

from (implicit) 2^{1000} to 583 [Bonnet, Kwon, Wood '22],

to 183 [Jacob, Pilipczuk '22], to 37 [Bekos, Da Lozzo, Hlineny, Kaufmann '22],

to 8 [Hlineny, Jedelsky '22].
A simple proof for 11 to be presented tomorrow.

Exists a planar graph with twin-width 7 [Kral, Lamaison '22].

Euler genus g

$2^{18g+O(1)}$ to $18\sqrt{47g} + O(1)$ [Král, Pekárková, Storgel '23].

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This approach does not extend to minor-closed families in general.

Grid Minor Theorem for twin-width

Contraction on matrices

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & r & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & r & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & r & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Twin-width of a matrix

tww(M) ⩽ d if ∃ a contraction sequence from M to 1 x 1 consisting of matrices with red number ⩽ d

Twin-width of a matrix

delete one row, replace the inconsistent entries by "R"

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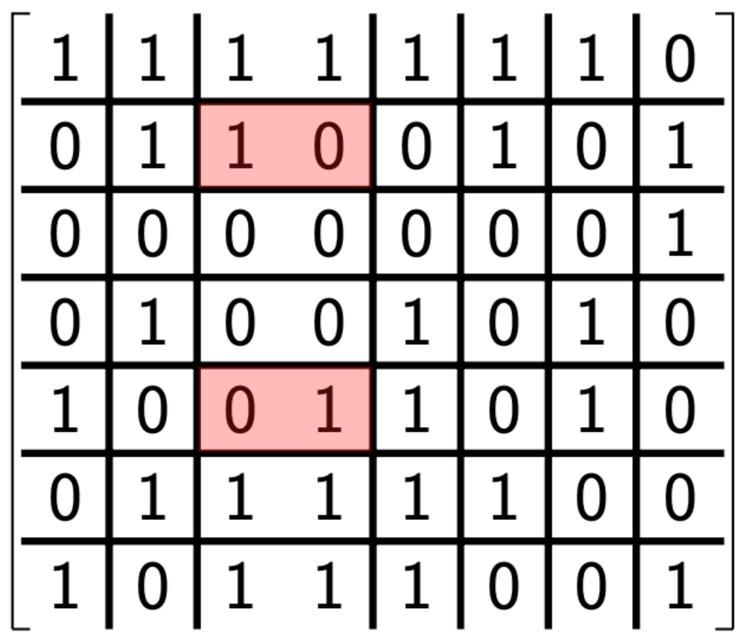
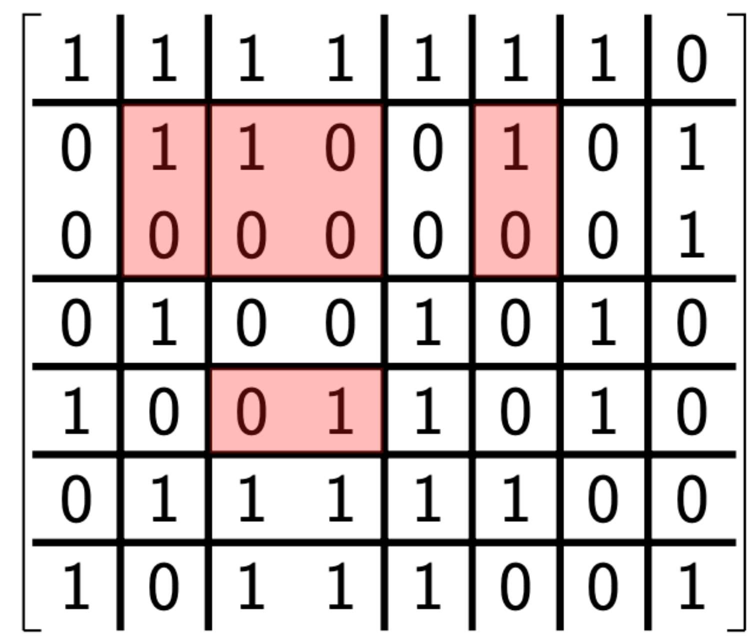
maximum number of “R”s over all rows and columns

Twin-width of a matrix

tww(M) $\leqslant$ d if $\exists$ a contraction sequence from M to 1 x 1 consisting of matrices with red number $\leqslant$ d

Partition viewpoint on matrices





Reorder columns and rows ⇝ we merge only consecutive rows / columns

call it “twin-ordered” matrix

Merging rows ⇔ “coarsening” row division by merging two row parts

red entry ⇔ “cell” (row part ∩ column part) is not “constant”

Twin-width of a matrix

tww(M) ⩽ d if for some M' obtained by a reordering of columns and rows, ∃ a sequence of divisions from m x n-division of M' to 1 x 1-division with max error value ⩽ d

Twin-width of a matrix

merging two consecutive row or column parts:

a non-constant cell is marked in "ERROR"

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maximum number of “ERROR”s over all row and column parts

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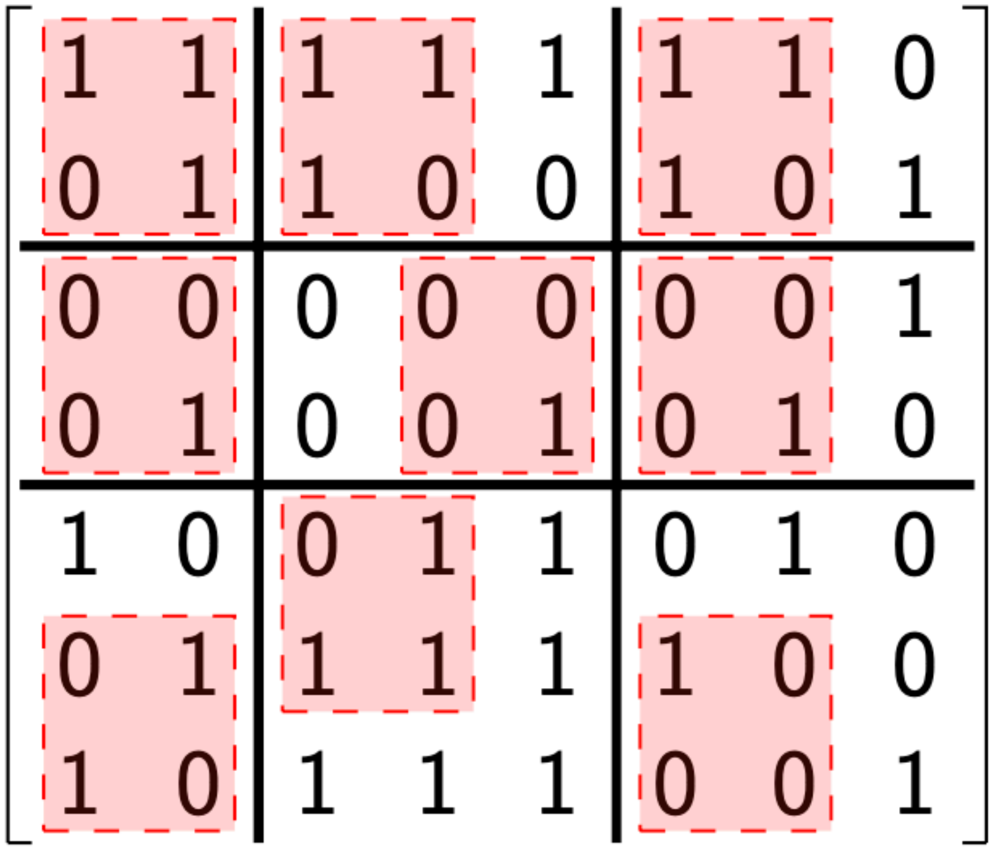
mixed minor



3-mixed minor = 3 x 3 division in which each cell is “mixed”

t-mixed free if M does not have t-mixed minor

Grid Theorem for Twin-width

[Bonnet, K. Thomassé, Watrigant 2020]

For a twin-ordered matrix M, we have

$$\frac{mxn(M) - 1}{2} \leq tww(M) \leq 2^{2^{O(mxn(M))}}$$

$mxn(M)$=largest size of a mixed minor

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For a twin-ordered matrix M, we have

$$\frac{mxn(M) - 1}{2} \leq tww(M) \leq 2^{2^{O(mxn(M))}}$$

$mxn(M)$=largest size of a mixed minor

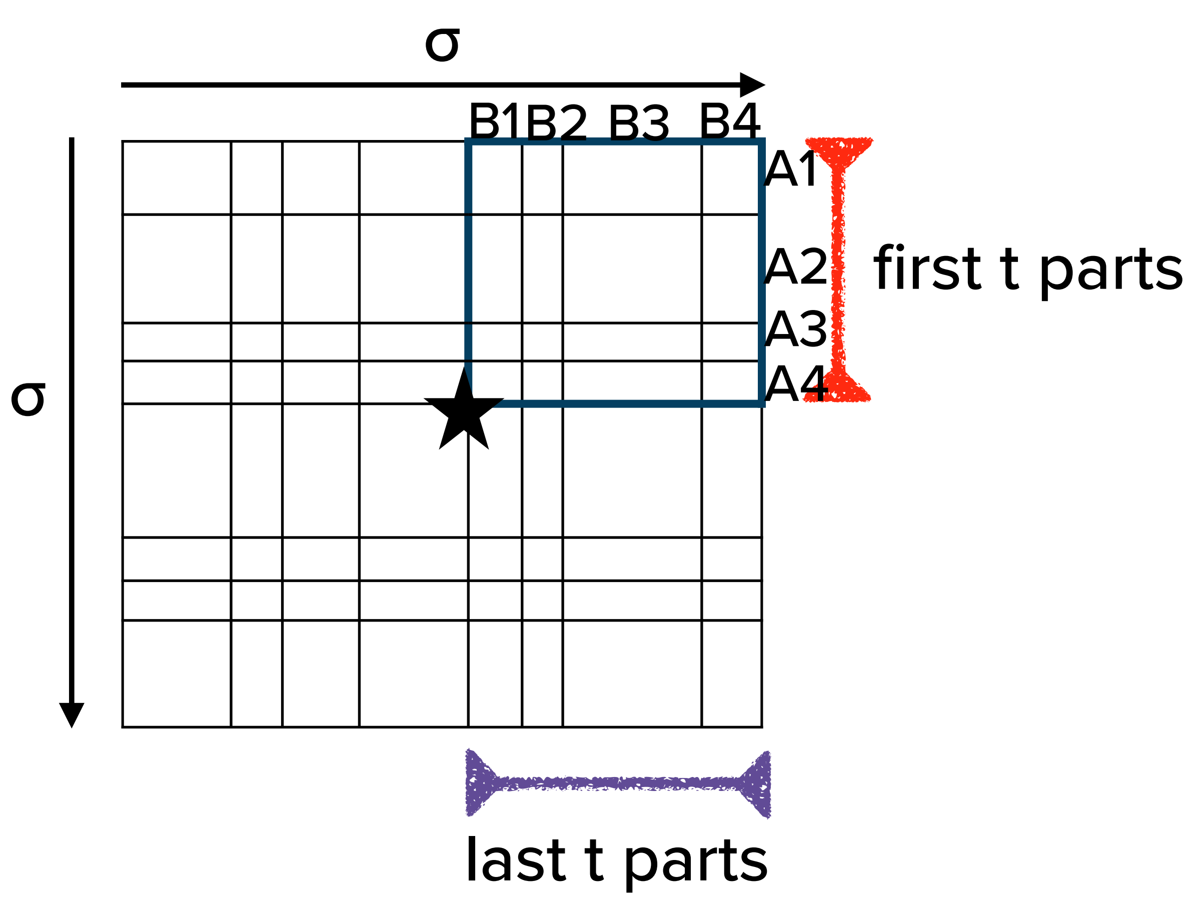
twin-width(G) is small

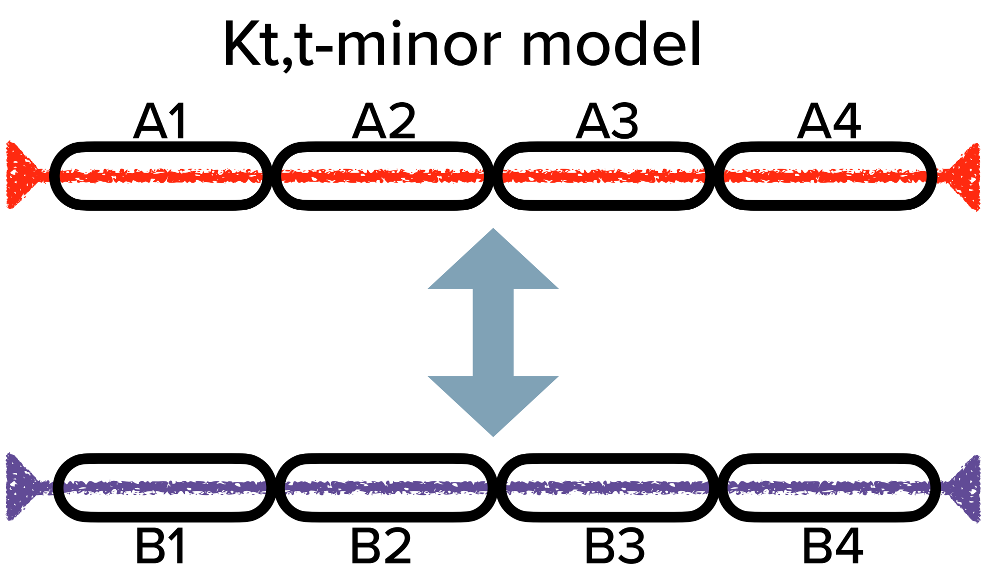
↕

there is a vertex ordering $\prec$ s.t.
$\text{adj}_{\prec}(G)$ does not have a large mixed minor.

Kt-minor-free graphs have bd tww

- If ∃ Hamiltonian path σ, A_σ has no 2t-mixed-minor; if it has…





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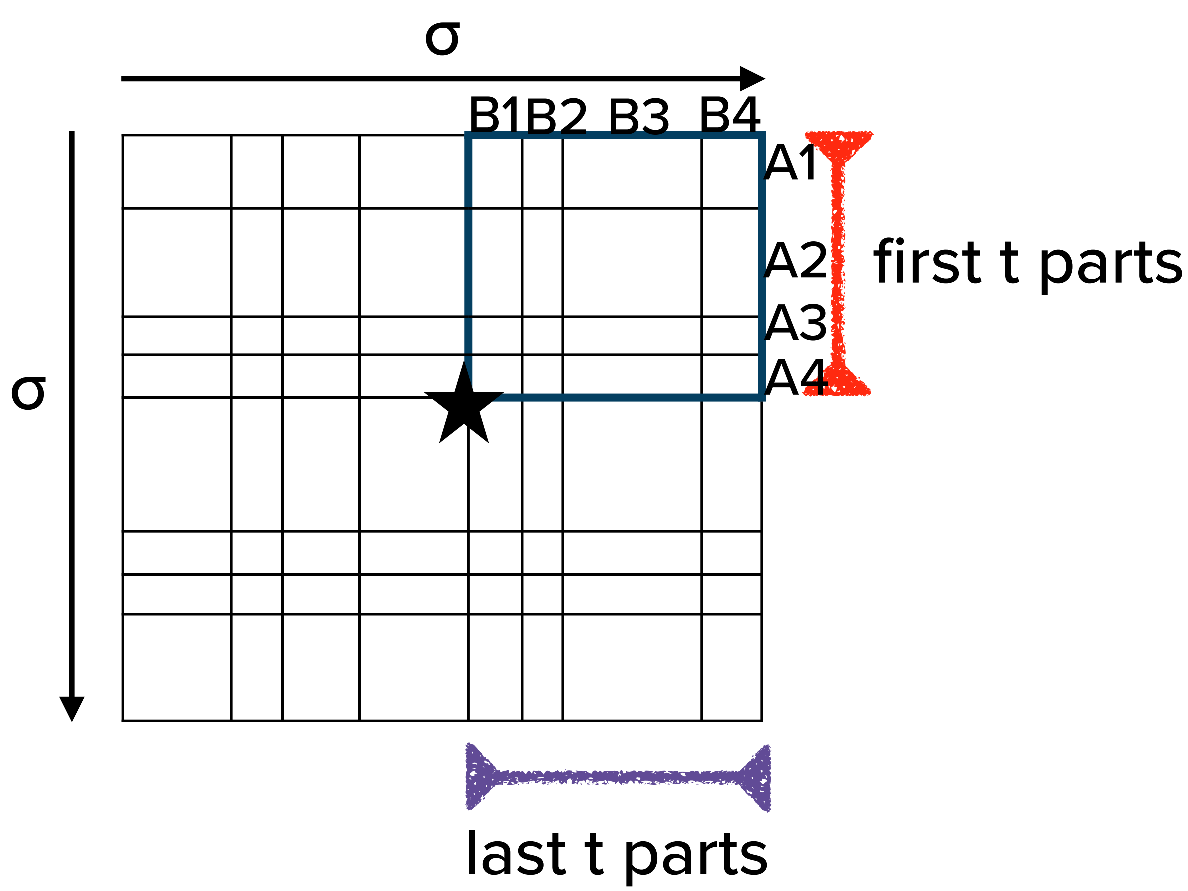
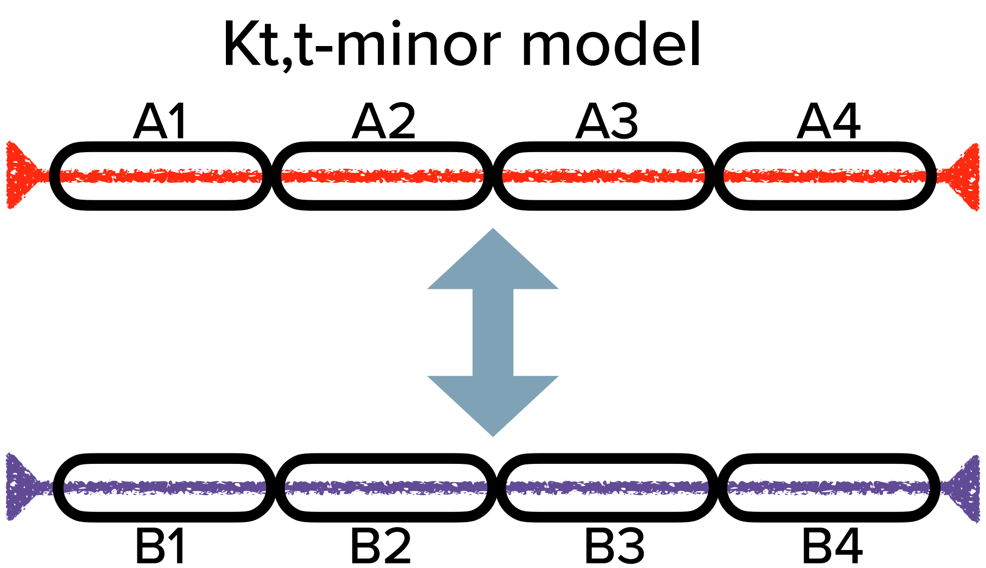
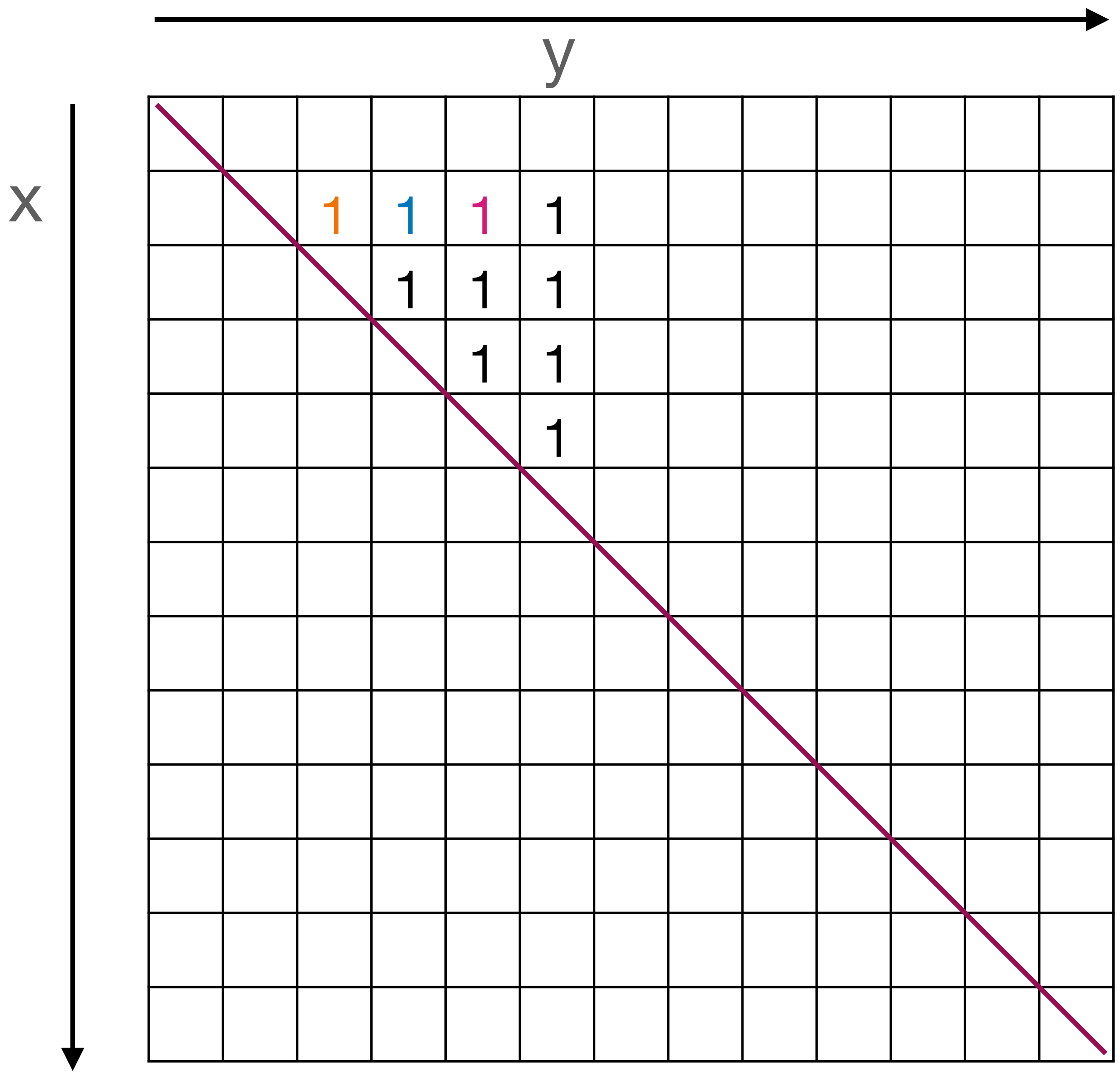




- General case can be proved using the **discovery order of Lex-DFS** as σ.

Unit Interval Graphs have bd tww

left-to-right ordering by the left endpoint of the unit interval



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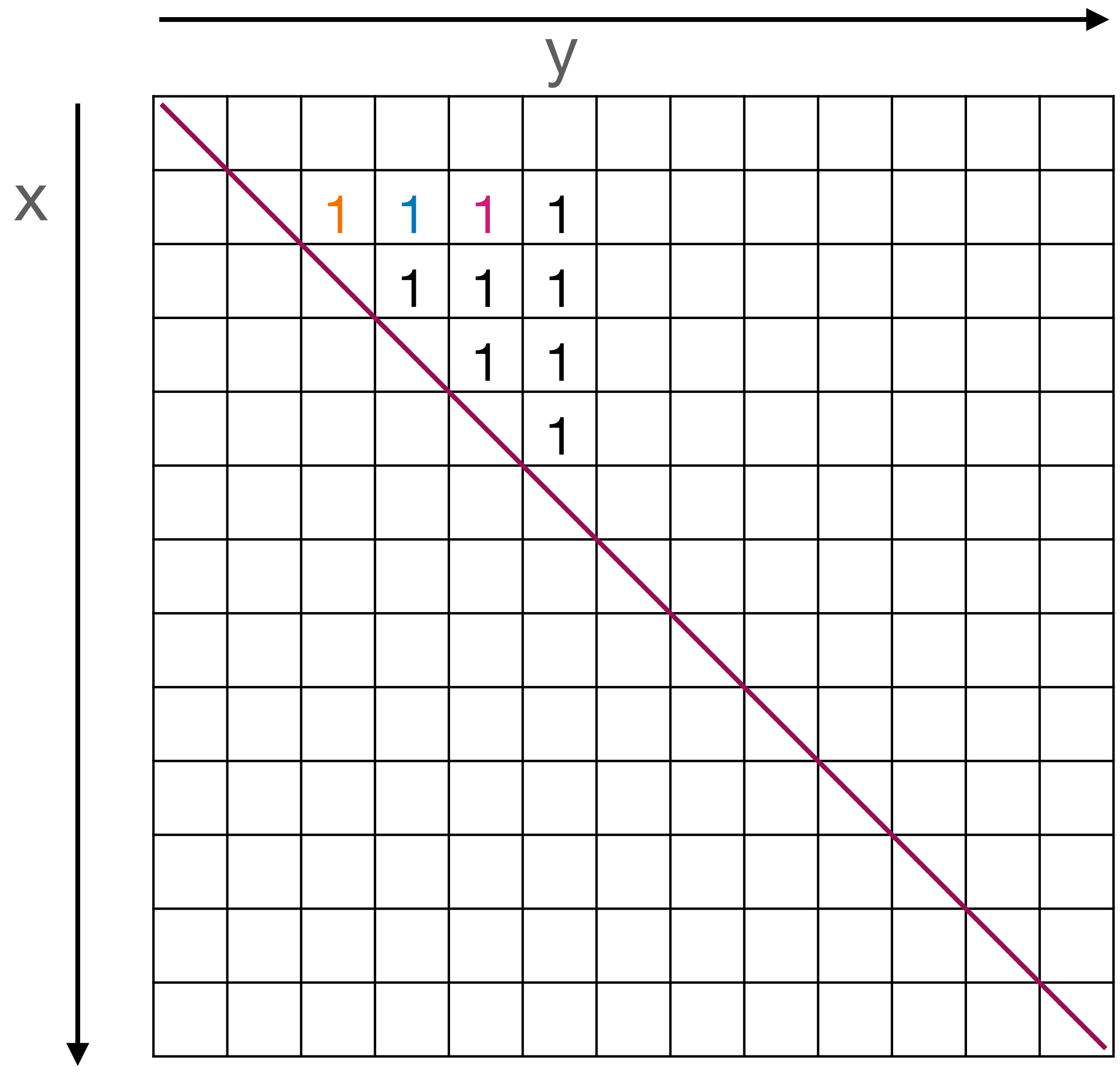
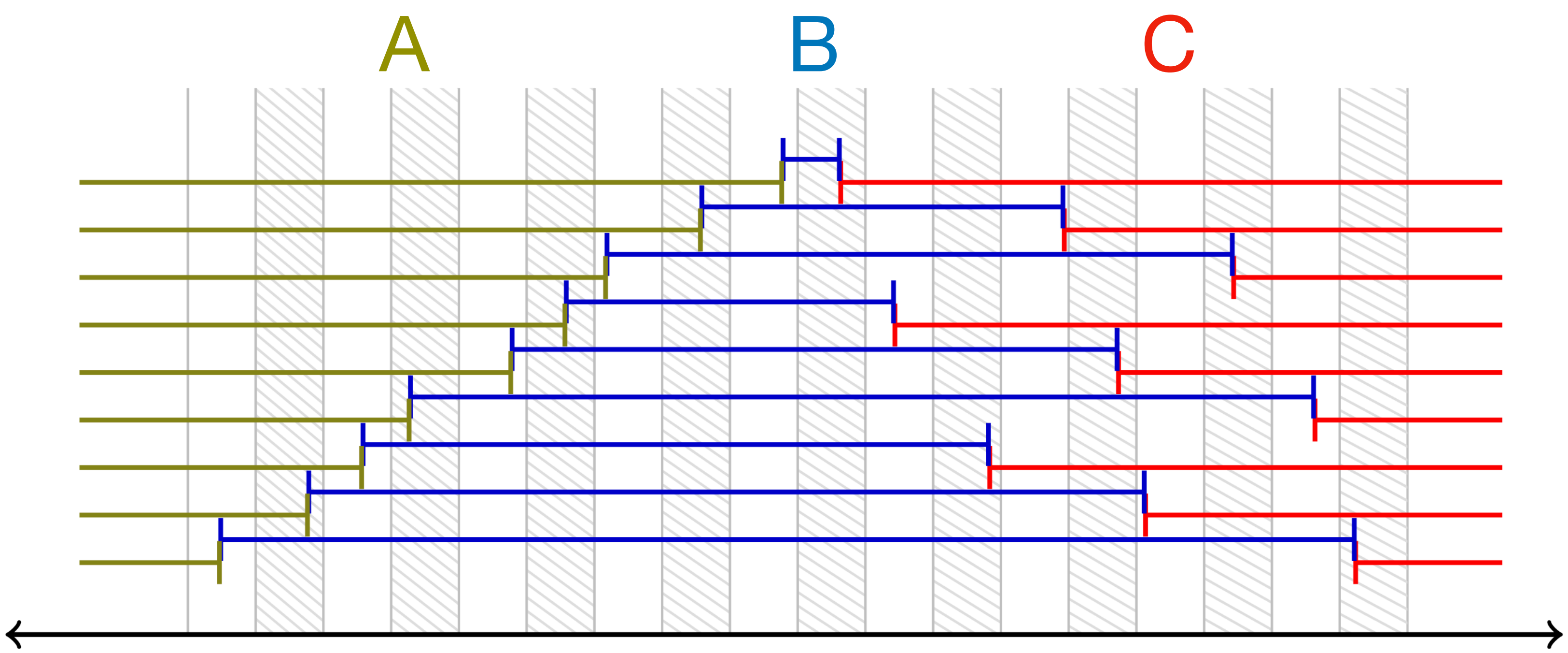
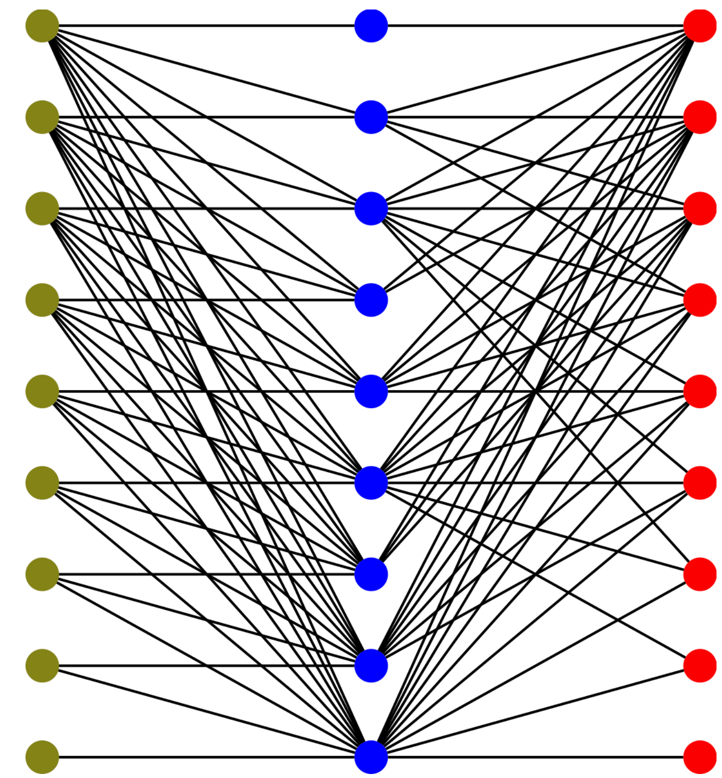


no 3-mixed grid

Interval Graphs have unbounded tww





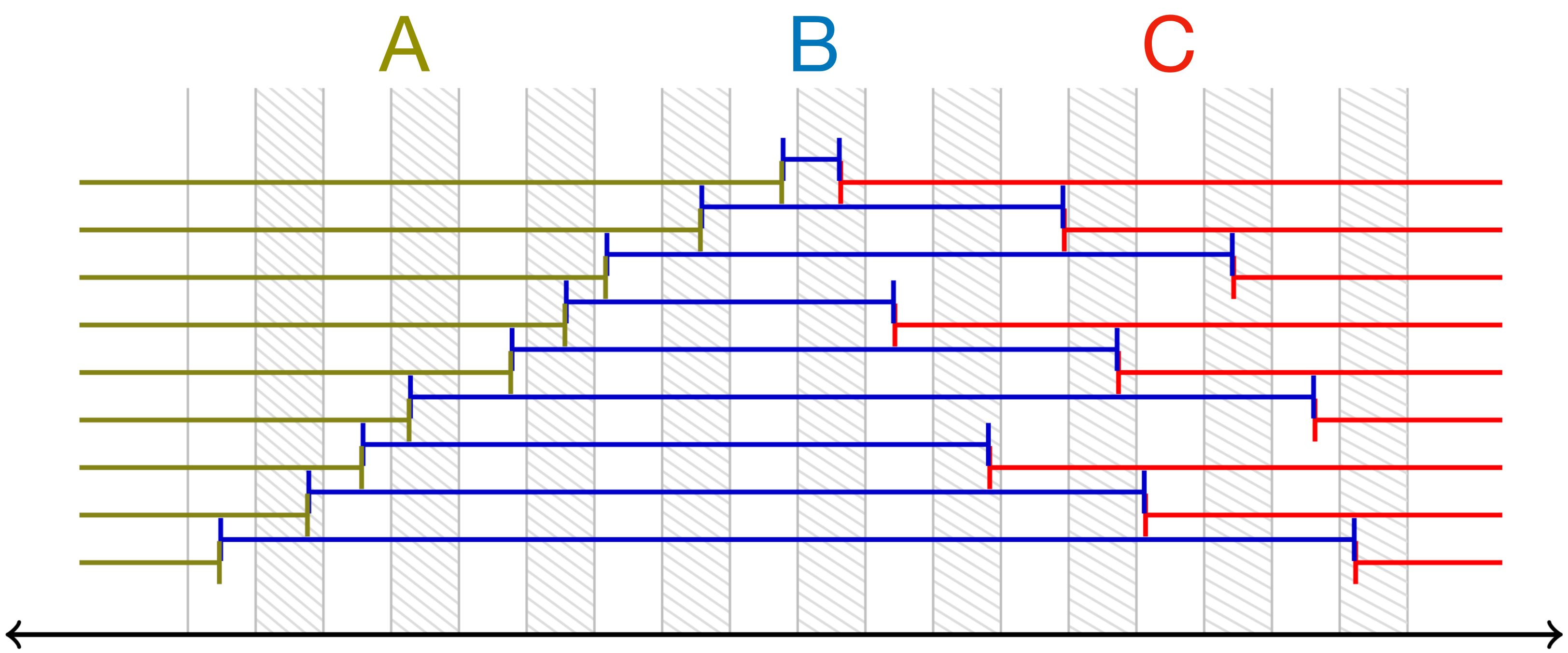
1								
			1					
						1		
	1							
				1				
							1	
		1						
					1			
								1

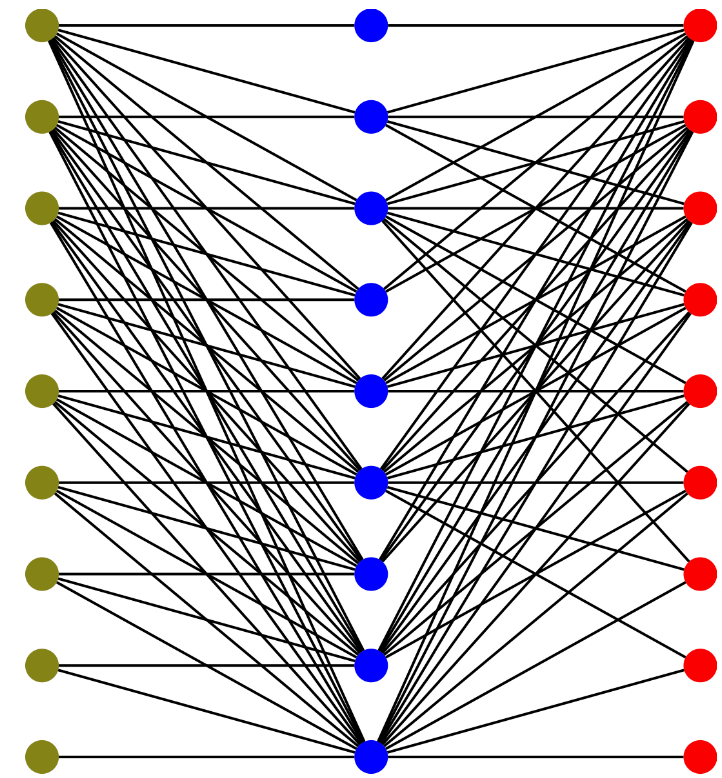
Can we use a different vertex order? Well…

The collection of all permutations are ‘encoded’ in the class of interval graphs.

The idea is formalized by the notion of ‘FO-interpretation/transduction’.

Interval Graphs have unbounded tww





1								
1	1	1	1					
1	1	1	1	1	1	1		
1	1							
1	1	1	1	1				
1	1	1	1	1	1	1	1	
1	1	1						
1	1	1	1	1	1			
1	1	1	1	1	1	1	1	1

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First-Order Model Checking

[Bonnet, K, Thomassé, Watrigant ’20]

FO model checking can be done in time $f(d,|\phi|)\cdot n$

when a d-contraction sequence is given.

[Bonnet, K, Thomassé, Watrigant '20]

Input: a graph G, first-order sentence ϕ.
Question: $G \models \phi$?

FO model checking can be done in time $f(d,|\phi|) \cdot n$

when a d-contraction sequence is given.

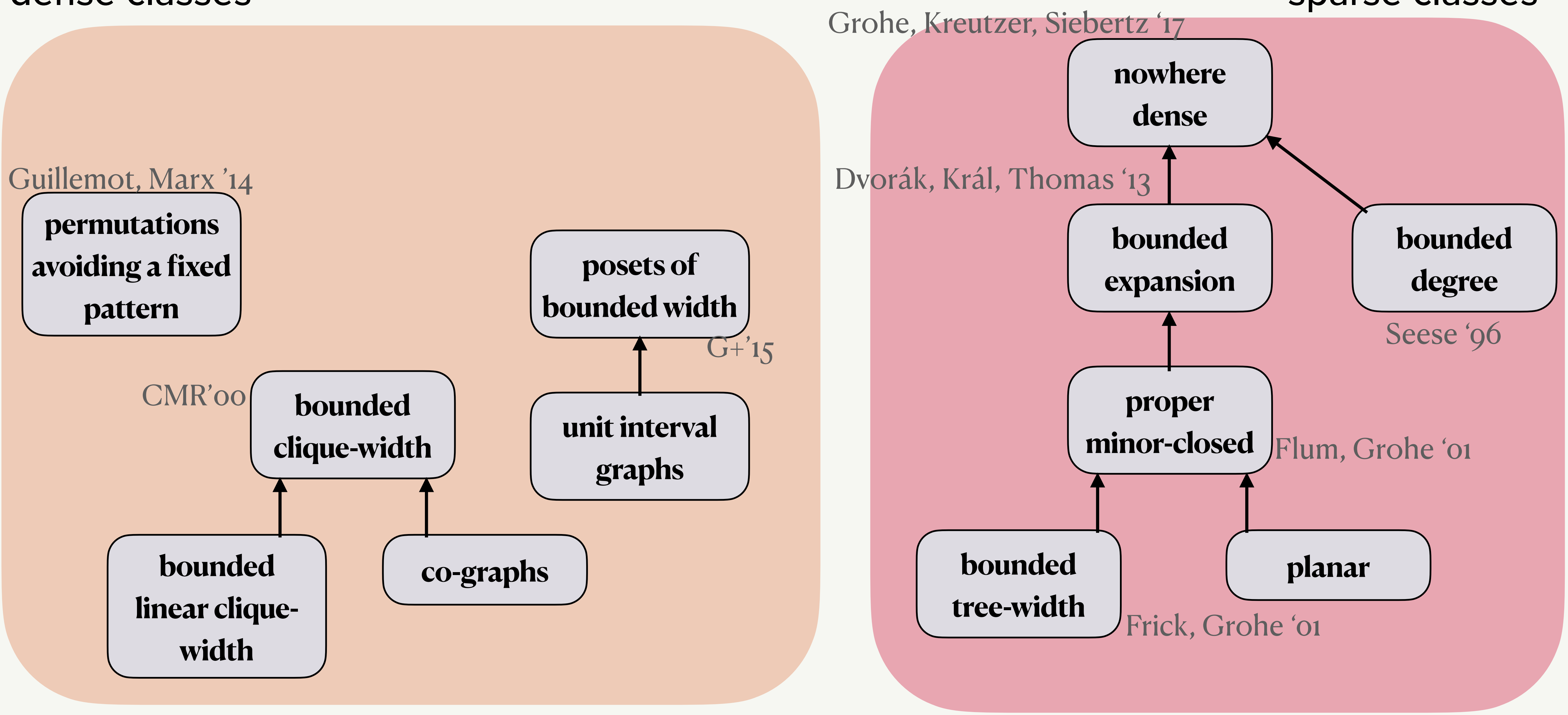
$$\Phi := \exists x_1 \exists x_2 \cdots \exists x_k \forall u \bigvee_{1 \leq i \leq k} \big((x_i = u) \vee E(x_i, u)\big)$$

$\rightsquigarrow G \models \Phi$ iff G has a dominating set of size k.

FO-model checking is FPT [BKTW'20]

dense classes

sparse classes



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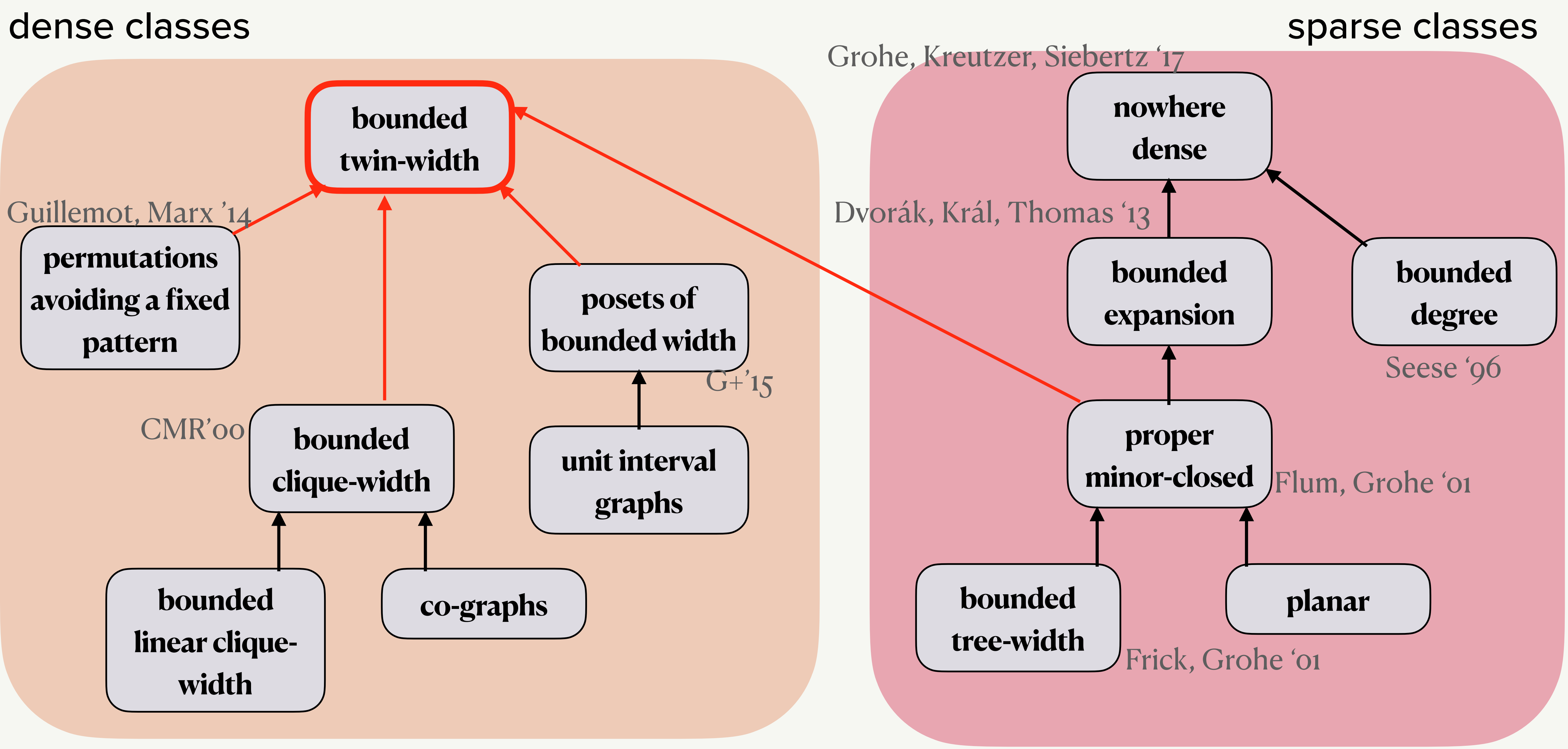

FO-transduction:

further extending the realm of twin-width

FO-interpretation: adding new relation via FO-logic

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$\tau : G = (V, E) \to$ Two-edge colored graph $(V, E \cup D)$
s.t. the new binary relation D is the set of
"all pairs of $V \times V$ satisfying an FO-formula $\varphi(x, y)$"

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- $\tau(x, y) := E(x, y) \vee \exists z(E(x, z) \wedge E(z, y))$; square
- $\tau(x, y) = \neg E(x, y)$; complement

FO-interpretation: adding new relation via FO-logic

$\tau : G = (V, E) \to$ Two-edge colored graph $(V, E \cup D)$
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$\tau(\mathscr{C}) = \{\tau(G) : G \in \mathscr{C}\}$

If $\mathscr{D} \subseteq \tau(\mathscr{C})$, "$\mathscr{C}$ (FO-)interprets $\mathscr{D}$"

FO-transduction:
FO-interpretation + introduce "unary relations"

Λ: G = (V,E) ➔ graph (V, S, E) for some vertex subset S

Now, you can query [$v \in S$].

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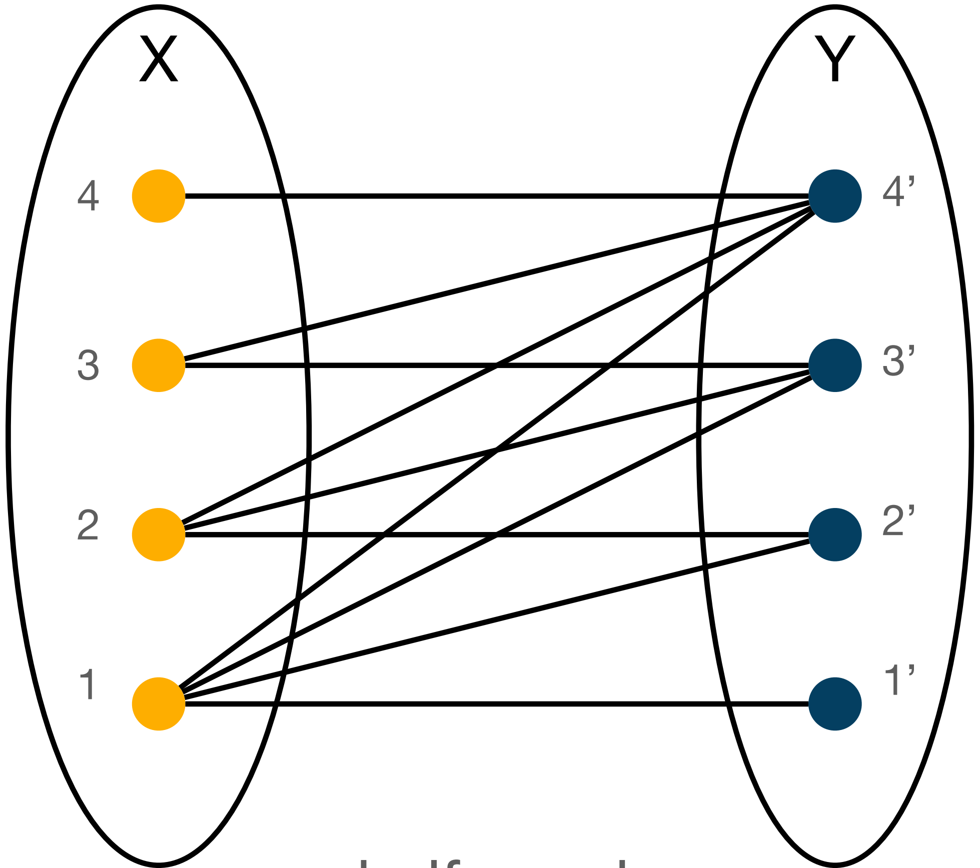


half-graph

- Linear order on the left set (i.e. transitive tournament)
- Color the right-hand side set by Y.
- $\varphi(a,b) := N(a) \cap Y \supset N(b) \cap Y$
- $R_\varphi = \{(1,2), (1,3), \cdots, (3,4)\}$

[Bonnet, K, Thomassé, Watrigant ’20]

Twin-width is stable under FO-transduction.

[Bonnet, K, Thomassé, Watrigant ’20]

Twin-width is stable under FO-transduction.

Read as: start from a graph class of bounded twin-width and apply an FO-transduction. The obtained class has bounded twin-width (depending on the first tww, and the transduction).

When twin-width is THE right measure

Permutation

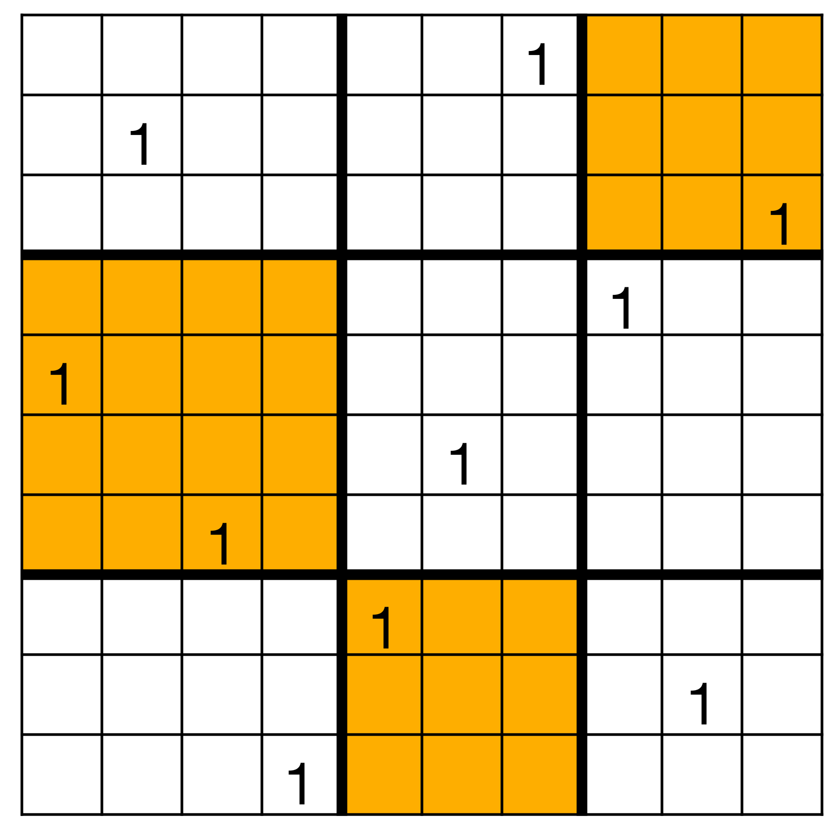
[BKTW'20] Let $\mathscr{C}$ be a hereditary class of permutations.
Either $\mathscr{C}$ is the class of all permutations, or
$\mathscr{C}$ avoids some pattern AND has bounded twin-width.

Suppose there exists a permutation $\sigma \notin \mathscr{C}$.

Then for every $\pi \in \mathscr{C}$, its matrix representation does NOT have $|\sigma|$-mixed minor.

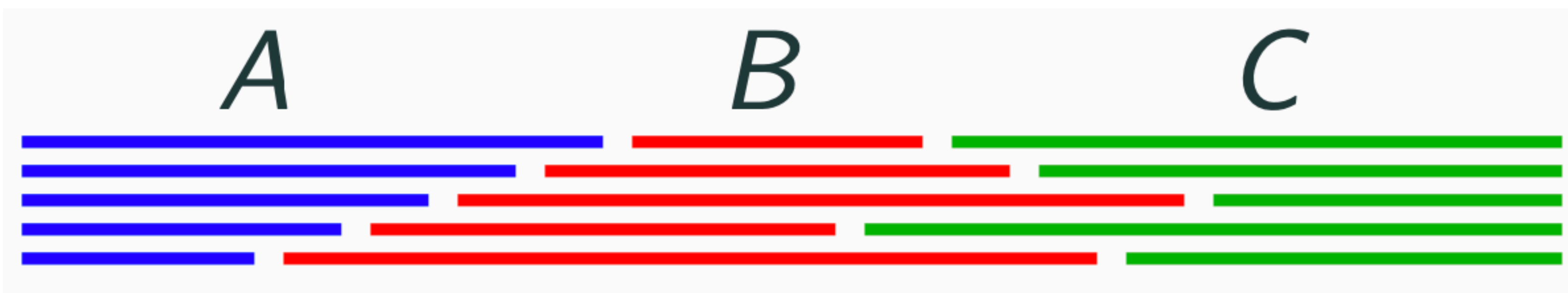
o/w, because $\mathscr{C}$ is hereditary, any permutation of length $|\sigma|$ - including σ itself - can be found as a sub-permutation, thus included in $\mathscr{C}$ due to hereditary property.

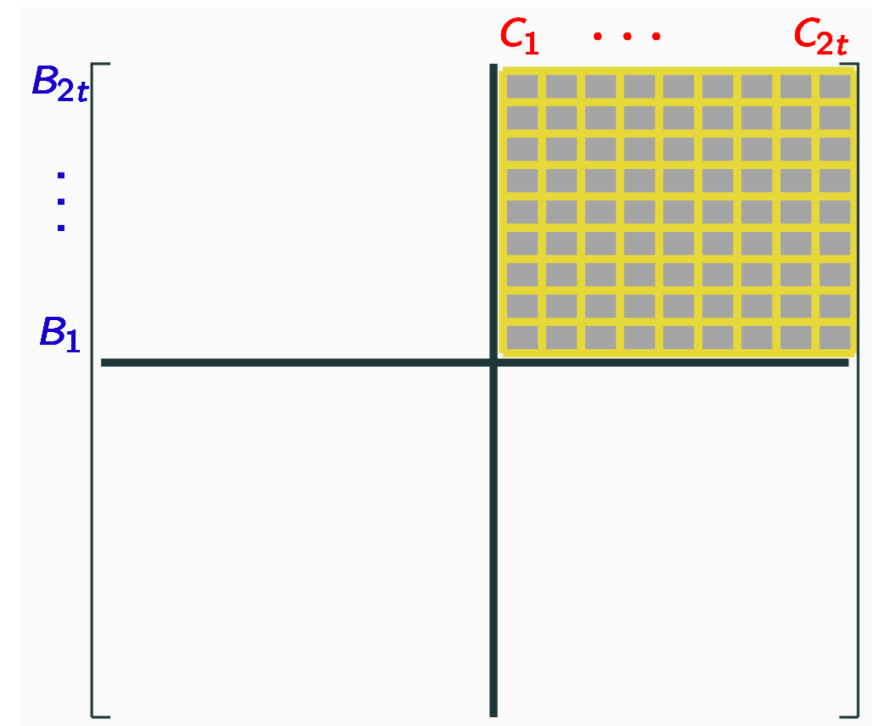
$\sigma = 312$



Interval Graph

$\prec$ = vertex ordering by lex order on the interval $(l(v), r(v))$





w.l.o.g. $B \prec C$

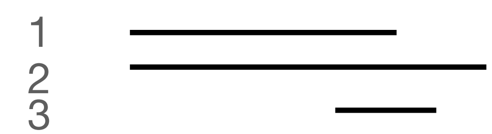
B

1	10	0	0	0
1	1	10	0	0
1	1	1	1	10
10	0	0	0	0
1	1	1	10	0

C

On B, we can interpret two different linear orders = permutation 23514

Interval Graph



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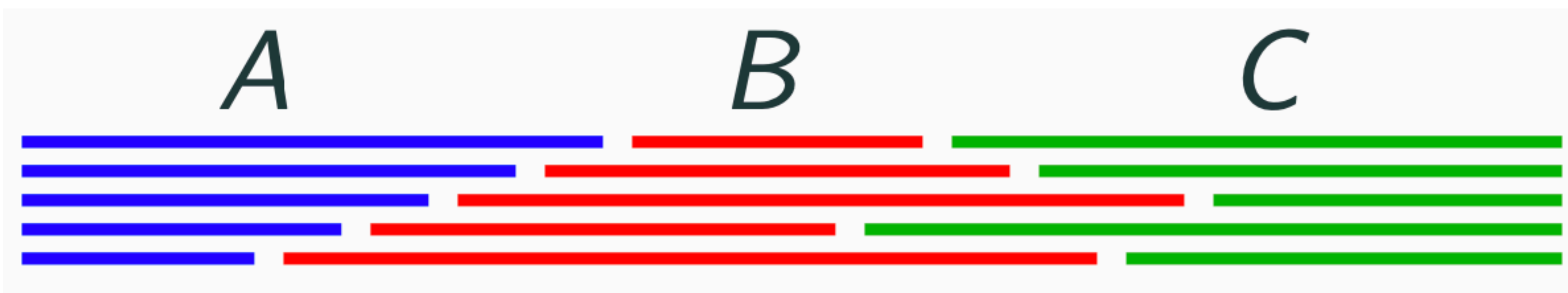

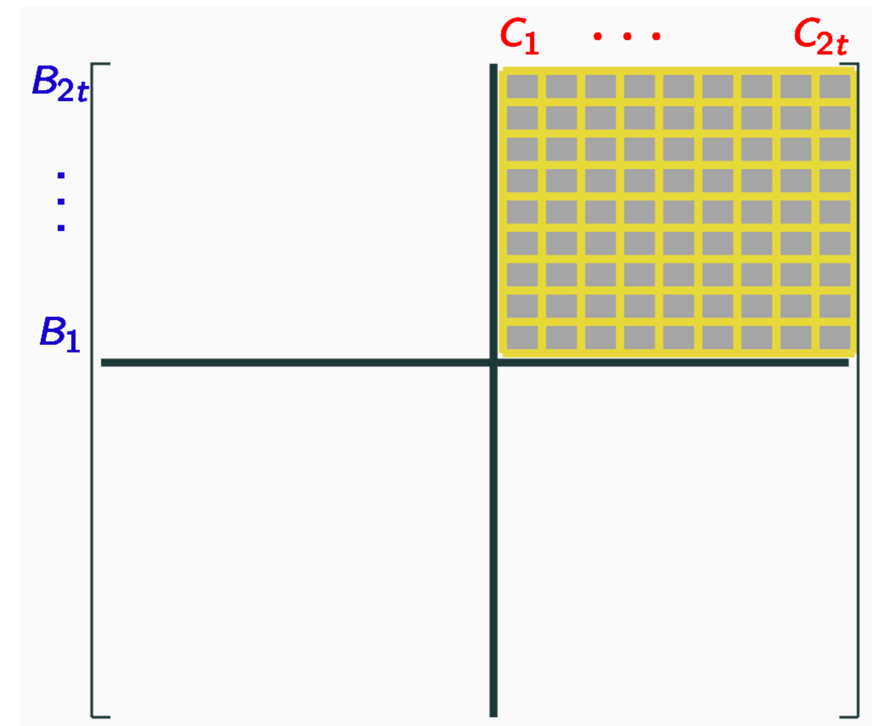

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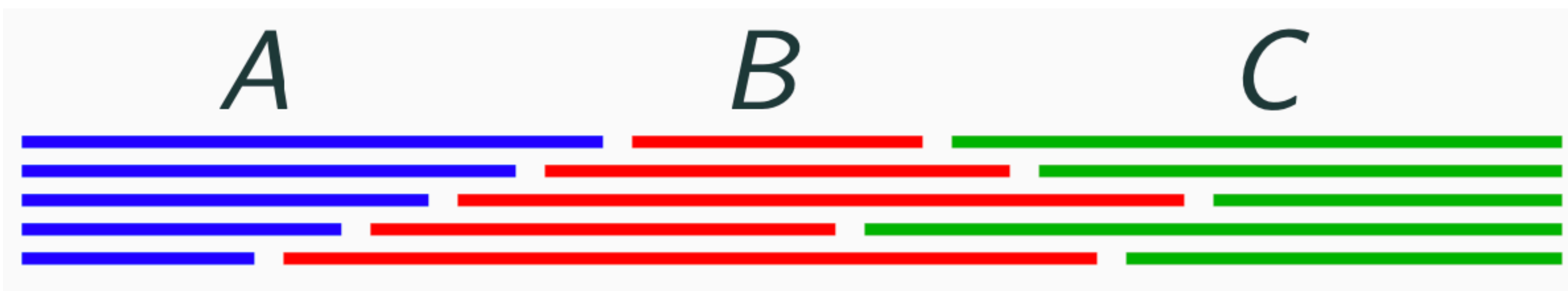

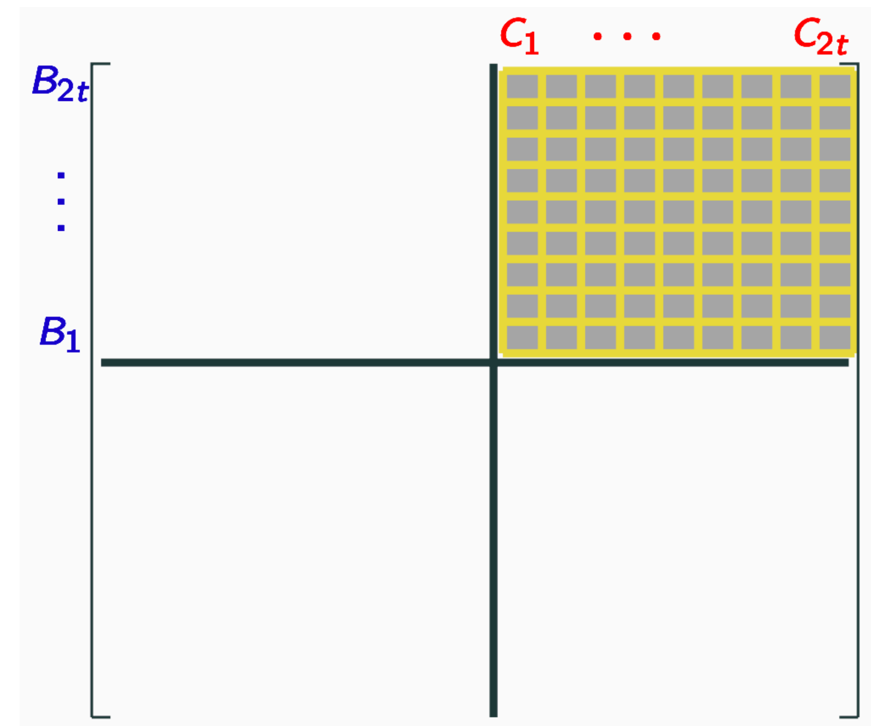

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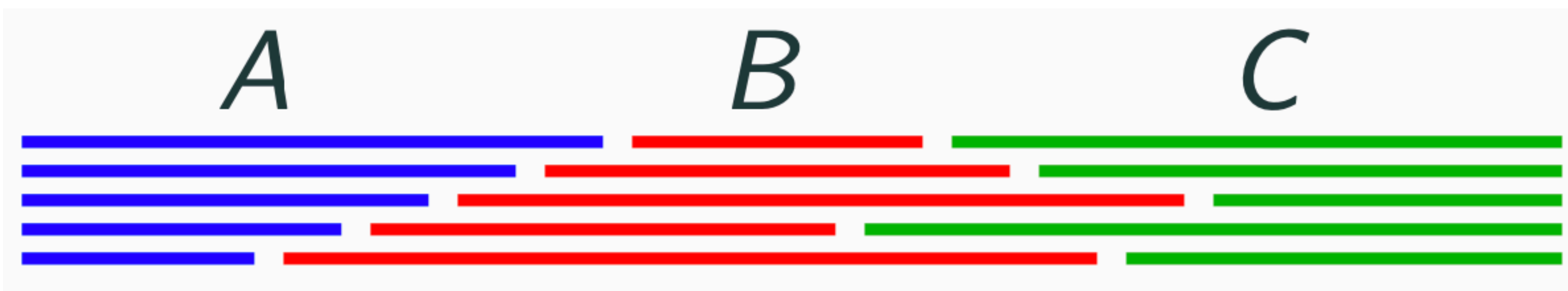

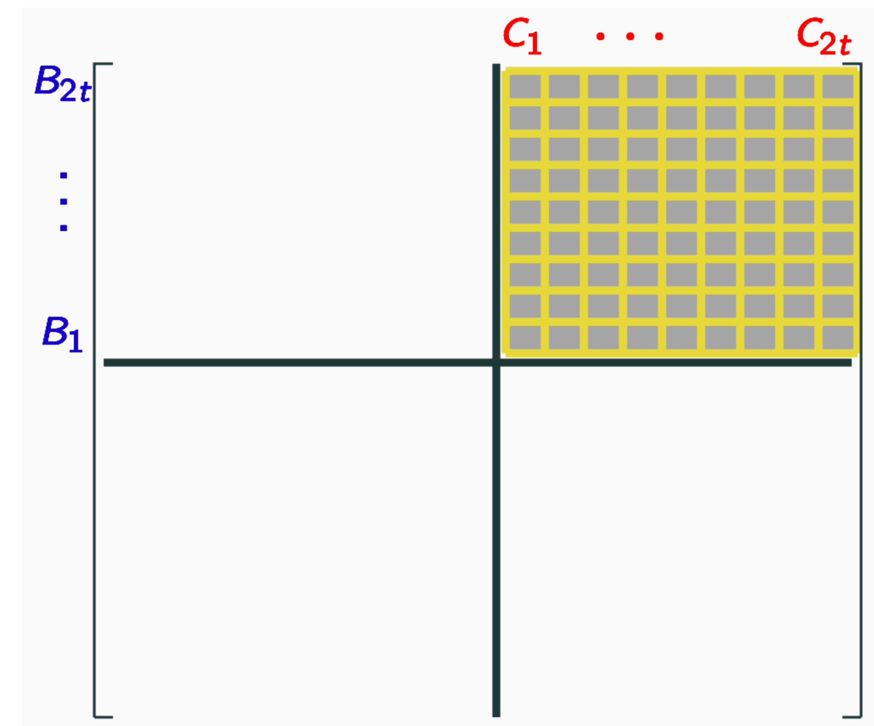

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C

w.l.o.g. $B \prec C$

If there is no upper bound on the mixed minor size of a hereditary class $\mathscr{C}$ of interval graphs, all permutations can be transduced from $\mathscr{C}$.

When twin-width is the right measure

[BKTW'20, BGOdSTT'21, HP'22, BCKKLT'22, GT'23]

The followings are equivalent (under some complexity assumption) for a hereditary class $\mathscr{C}$ consisting of **interval graphs | permutations | ordered graphs | tournaments | circle graphs | rooted directed path graphs.**

1. FO model-checking is FPT on $\mathscr{C}$.
2. $\mathscr{C}$ has bounded twin-width.
3. $\mathscr{C}$ does NOT FO-transduce the class of all graphs.
4. The growth of $\mathscr{C}$ is $2^{O(n)}$.

Unwinding a contraction sequence

$\chi(G) \leq (d+2)^{\omega-1}$ via unwinding

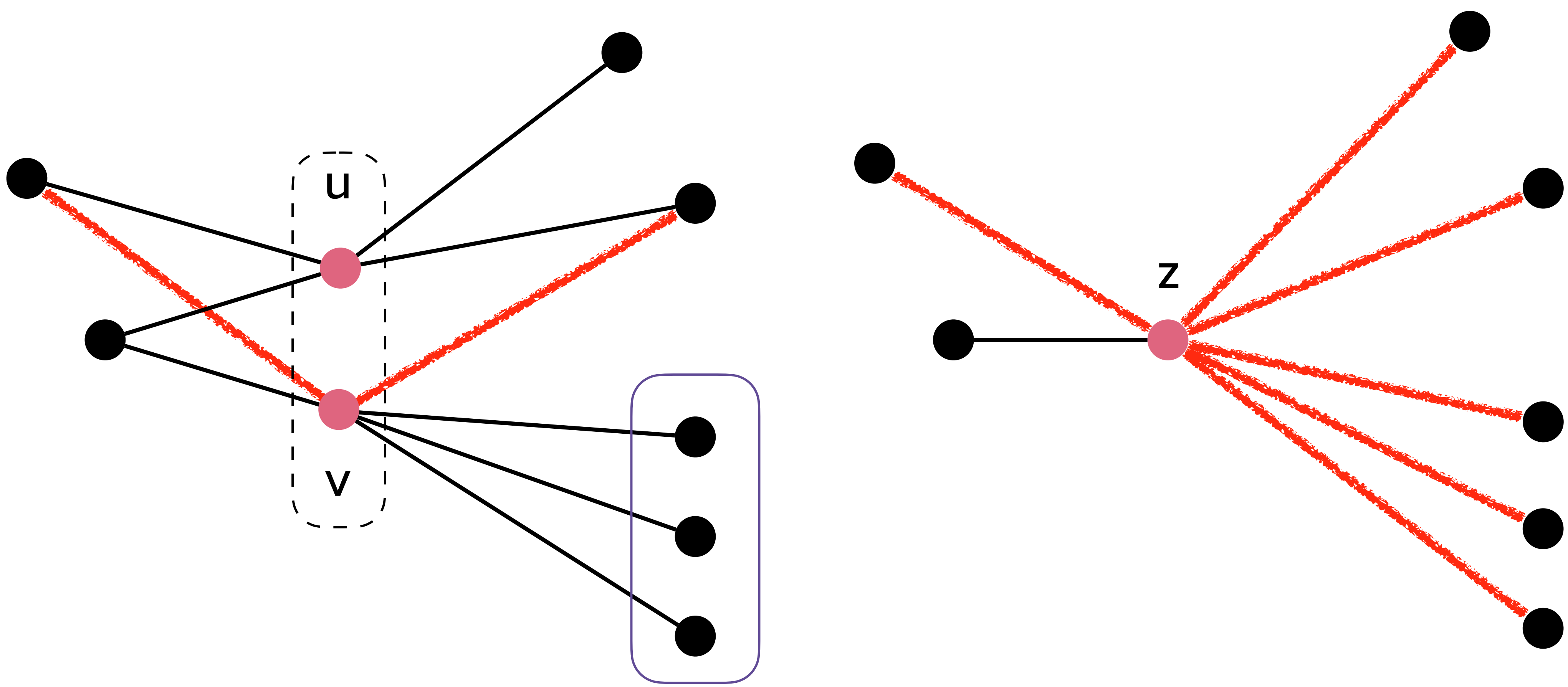
$\omega = 2$, i.e. triangle-free G.

Consider the contraction sequence $G_n, \ldots, G_{i+1}, G_i, \ldots, G_1$ backwardly.

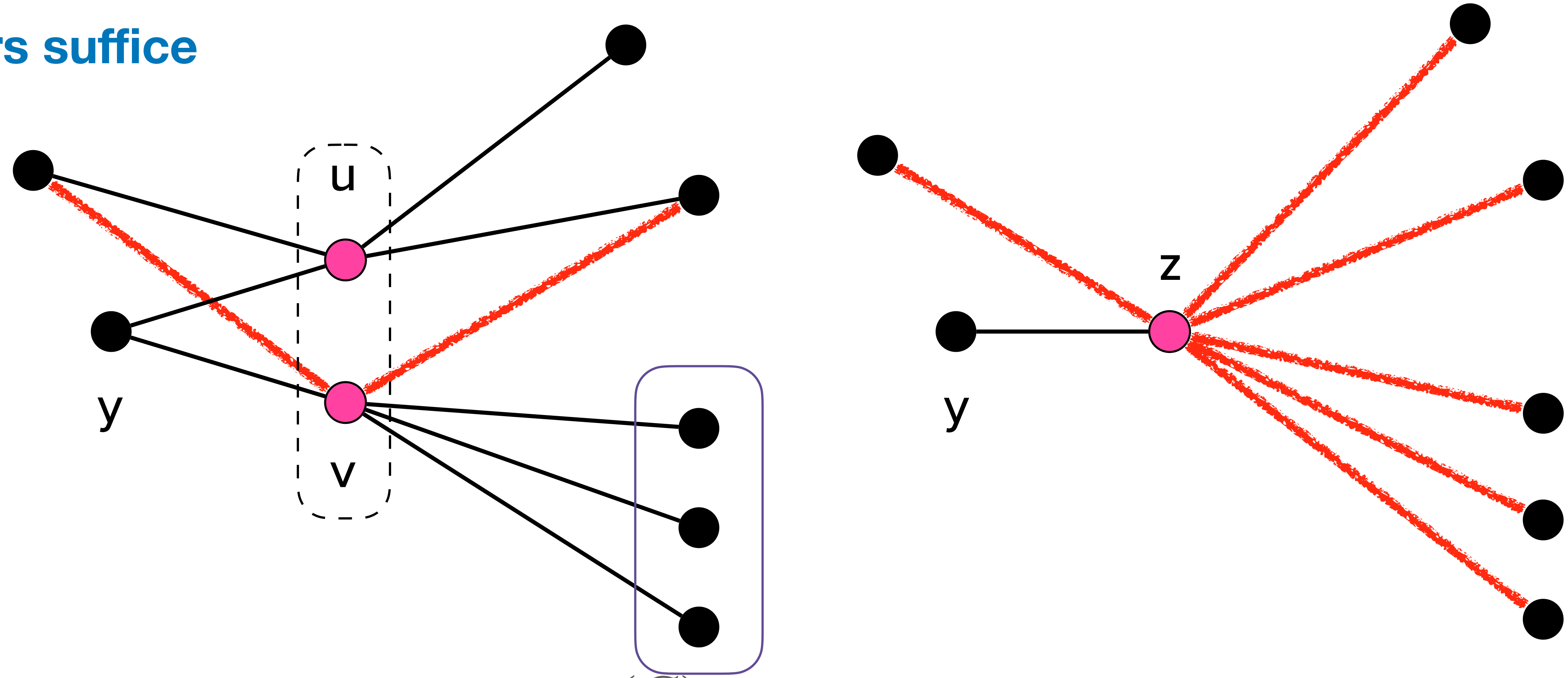
u inherits the color of z. Let's decide the color of v.

$c(v) = c(z)$ if (u, v) is non-adjacent in G_{i+1} ; proper coloring

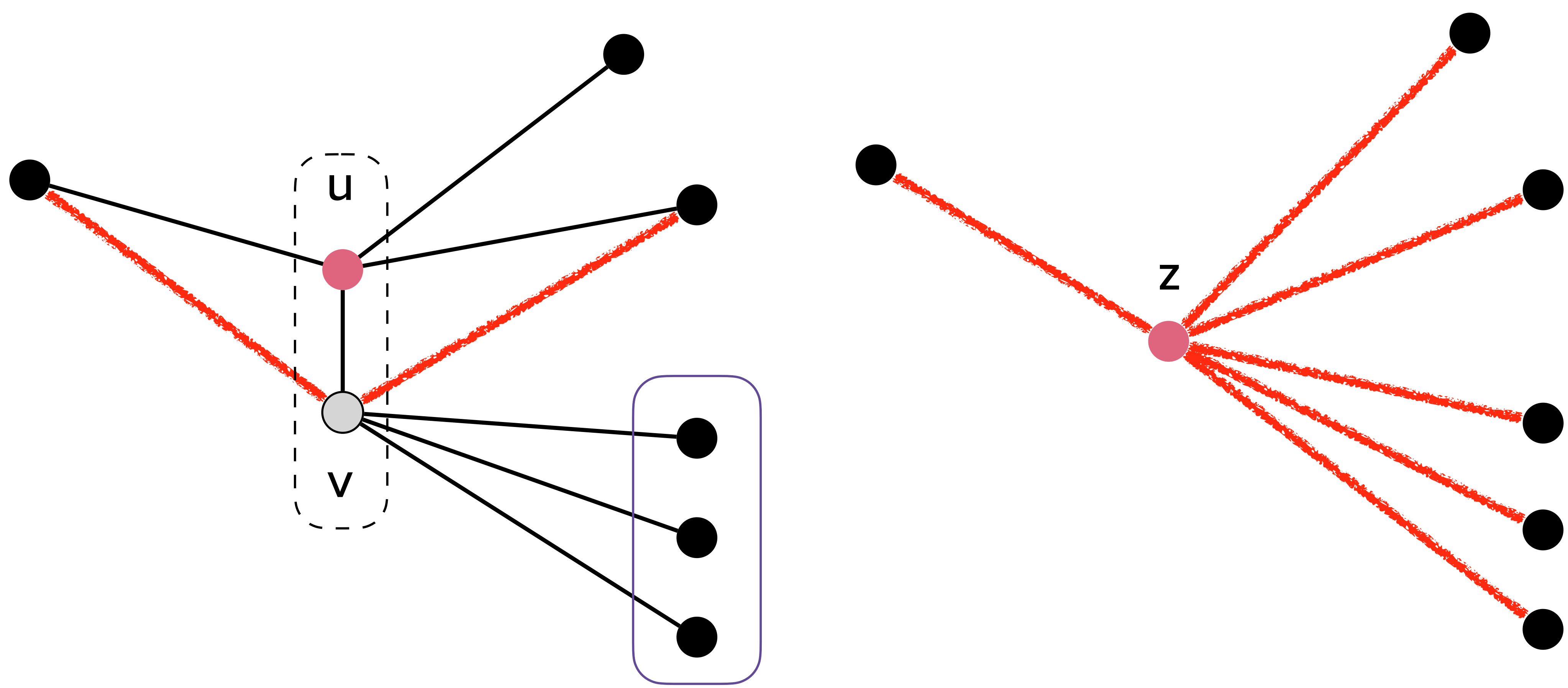
v gets the smallest available color if (u, v) is black/red-adjacent in G_{i+1}



d+2 colors suffice



z incident with a black edge → $z(G)$ independent → u and v non-adjacent in G_{i+1}



z incident with red edges only → v has black+red degree $\leq$ d+1 in G_{i+1}

χ-bounding function for twin-width

[Bonnet, Geniet, Kim, Thomassé, Watrigant '21] χ-bounded.

[Pilipczuk, Sokołowski '22] χ-bounded by quasi-polynomial.

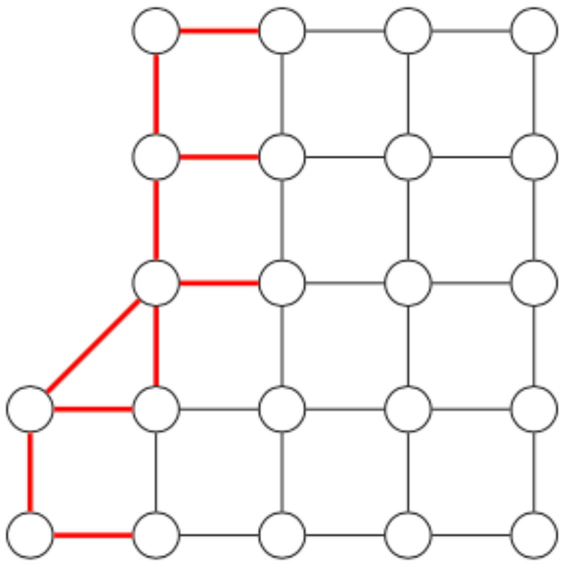
[Bourneuf, Thomassé '23] χ-bounded by polynomial.

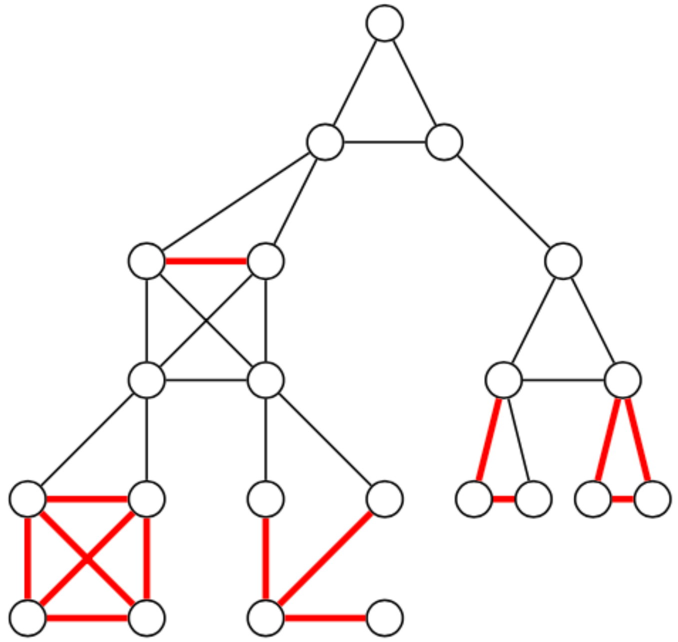
[Gajarský, Pilipczuk, Toruńczyk] linearly χ-bounded when sparse.

Twisting twin-width

Clique-width via contraction sequence

... s.t. any red component has bounded size





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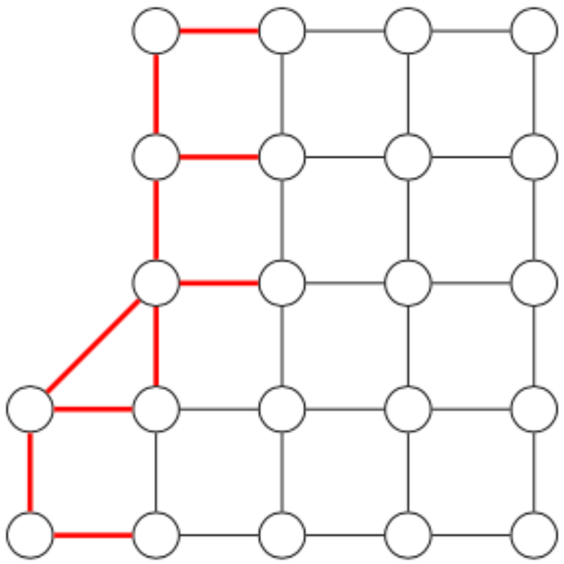
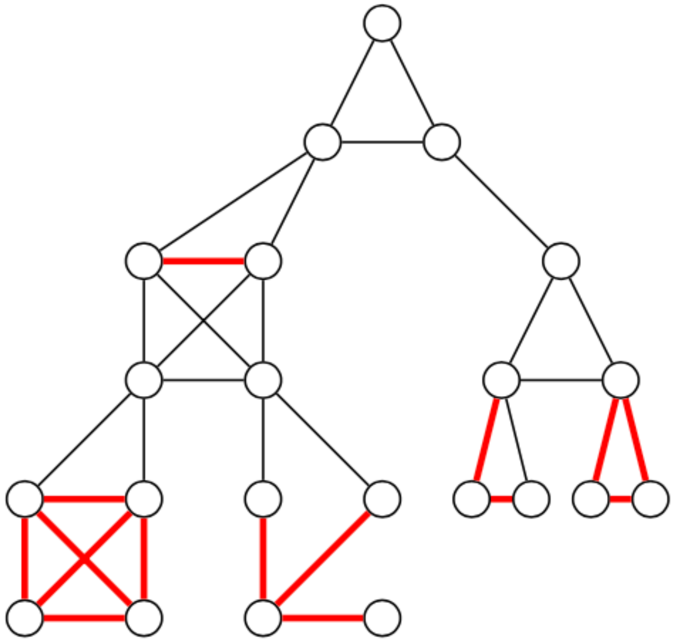
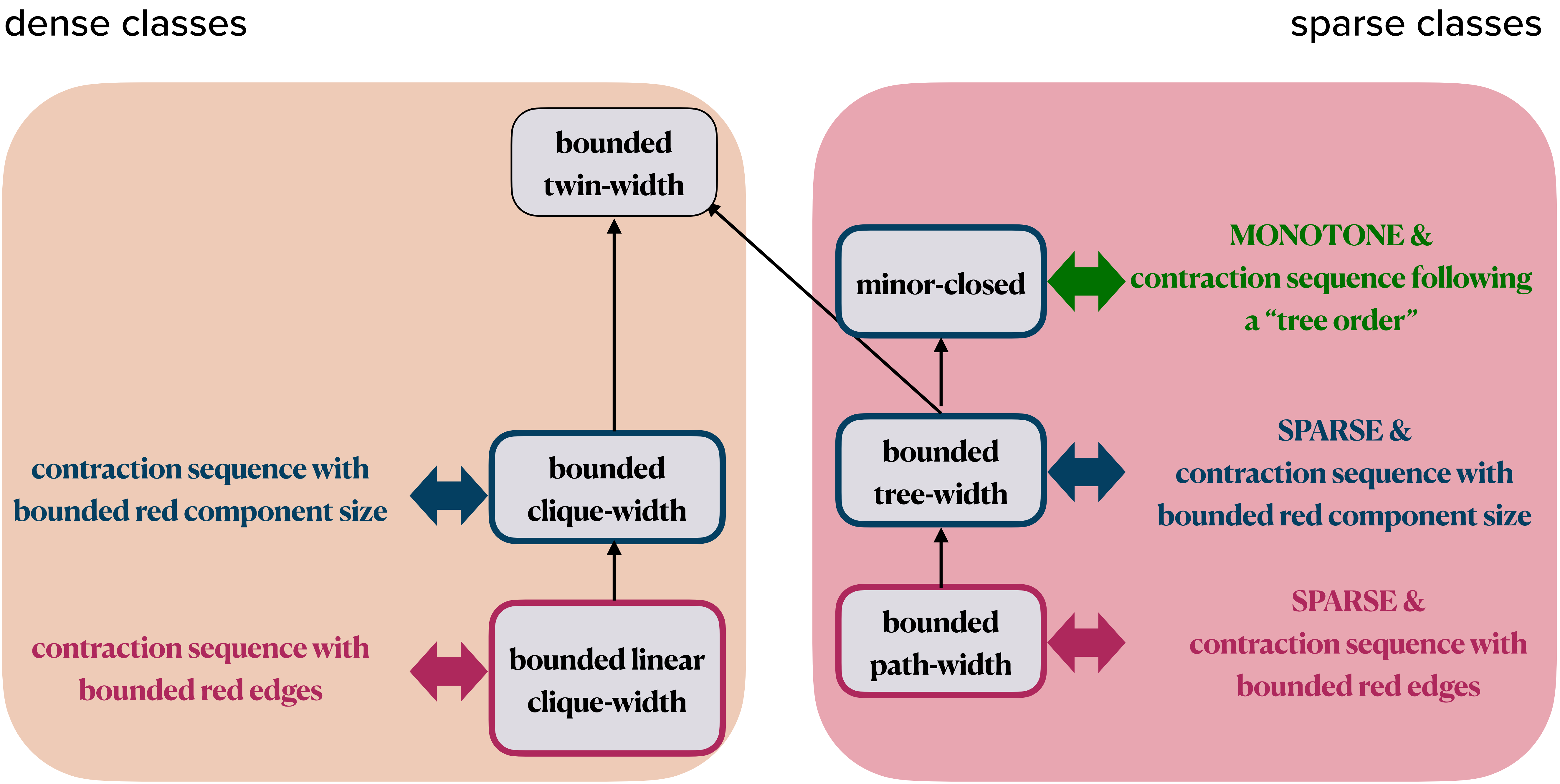




A graph class C has bounded clique-width
if and only if

C has bounded component twin-width

Characterization via twin-width' friends



[Bonnet, Kim, Reinald, Thomassé 2022]

Concluding Remarks

- Other cool tools not covered here, leading to applications in logic, data structure, labeling scheme, structural insights, etc.
- We still do not know how to compute f(d)-contraction sequence when the input has tww d in FPT, even in XP time.
- Twin-width for non-binary relation, e.g. hypergraphs?
- Explicit construction of cubic graphs of unbounded twin-width.
- O(1)-approximation for Max Independent Set on bounded tww? (implies PTAS)

Thank you!