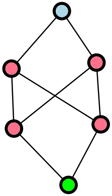
χ-Boundedness for posets

Gwenaël Joret

Université libre de Bruxelles

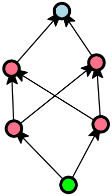
joint work with Piotr Micek, Michał Pilipczuk, and Bartosz Walczak

Drawing posets



diagram

Drawing posets



diagram

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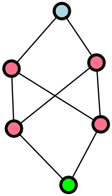
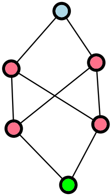
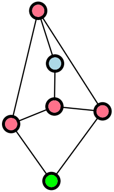


diagram

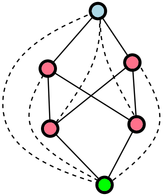
Drawing posets



diagram



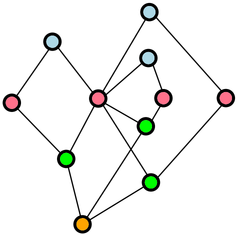
cover graph



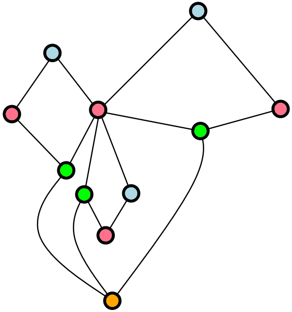
comparability graph

Drawing posets

poset with ...

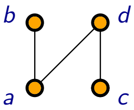


non-planar diagram

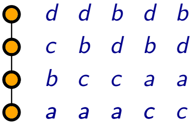


planar cover graph

Dimension



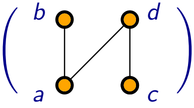
poset **P**



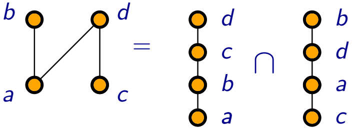
linear extensions of **P**

Dimension of **P** is the minimum d such that there are d linear extensions $L_1, \dots, L_d$ of **P** with

$$\mathbf{P} = \bigcap_{i \in [d]} L_i$$

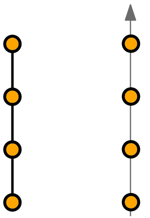
dim 

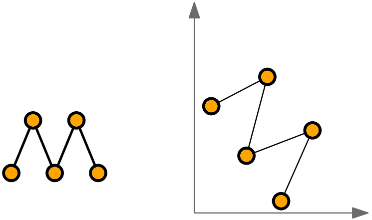
$\leqslant 2$ as



Dimension: Geometric view

Dimension of **P** is the least d such that **P** is isomorphic to a subposet of $\mathbb{R}^d$



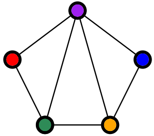


Why dimension?

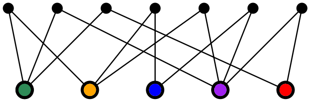
A natural notion...

...with interesting connections, e.g.:

Incidence posets:



G

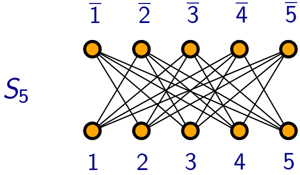


$\mathbf{P}_G$

Schnyder 1989

G planar $\Leftrightarrow \dim(\mathbf{P}_G) \leqslant 3$

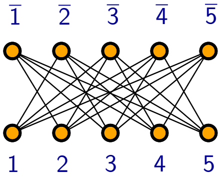
Standard examples have large dimension



$$\dim(S_k) = k$$

Standard examples have large dimension

S_5



$$\dim(S_k) = k$$

If **P** contains S_k then $\dim(\mathbf{P}) \geqslant k$

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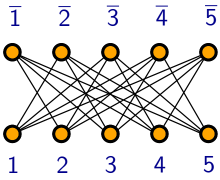

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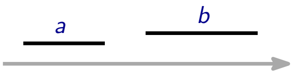
If **P** contains S_k then $\dim(\mathbf{P}) \geqslant k$

Standard example number $\mathrm{se}(\mathbf{P})$: Largest k s.t. **P** contains S_k

$\dim(P) \geqslant \mathrm{se}(P)$

Posets with large dimension but no big standard example

Interval orders



Interval orders $\equiv$ posets with no S_2

Universal interval order U_n: all intervals $[i, j]$ with $1 \leqslant i < j \leqslant n$.

$\dim(U_n) \geqslant \log_2 \log_2 n$

χ-Boundedness and dim-boundedness

chromatic number $\leftrightsquigarrow$ dimension

cliques $\leftrightsquigarrow$ standard examples

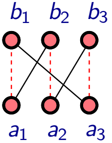
Family $\mathcal{G}$ of graphs is **χ-bounded** if there exists $f : \mathbb{R} \to \mathbb{R}$ s.t. $\chi(G) \leqslant f(\omega(G))$ for all $G \in \mathcal{G}$

Family $\mathcal{P}$ of posets is **dim-bounded** if there exists $f : \mathbb{R} \to \mathbb{R}$ s.t. $\dim(\mathbf{P}) \leqslant f(\mathrm{se}(\mathbf{P}))$ for all $\mathbf{P} \in \mathcal{P}$

Dimension as a hypergraph coloring problem

Dimension $=$ least number of linear extensions reversing all incomparable pairs (a, b)

Alternating cycle: Incomparable pairs $(a_1, b_1), \ldots, (a_k, b_k)$ s.t. $a_i \leqslant_P b_{i+1}$ $\forall i$ (cyclically)



Lemma: Set I of incomparable pairs can be reversed with one linear extension $\Leftrightarrow$ I has no alternating cycle

Hypergraph $\mathcal{H}$:

- vertex set $= \{$ incomparable pairs $\}$
- hyperedges $\leftrightarrow$ alternating cycles
- $\chi(\mathcal{H}) = \dim(\mathbf{P})$

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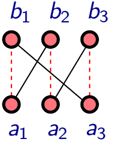

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- vertex set = $\{$ incomparable pairs $\}$
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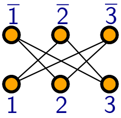
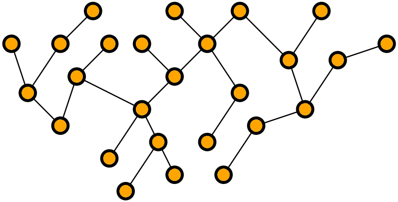
- $\chi(\mathcal{H}) = \dim(\mathbf{P})$
- cliques $\leftrightarrow$ standard examples
- $\omega(\mathcal{H}) = \mathrm{se}(\mathbf{P})$



What is this talk about?

"If a poset is nice then its dimension is small"



Trotter and Moore 1977

If cover graph of **P** is a forest then $\dim(\mathbf{P}) \leqslant 3$

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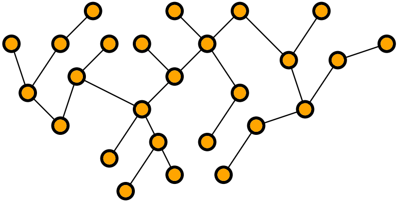

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Felsner, Trotter, Wiechert 2015

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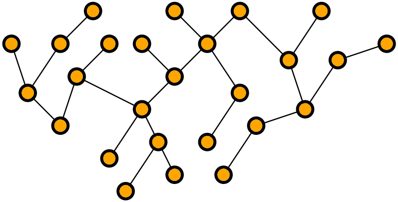

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J., Micek, Trotter, Wang, Wiechert 2014

If cover graph of **P** has treewidth $\leqslant 2$ then $\dim(\mathbf{P}) \leqslant 1276$

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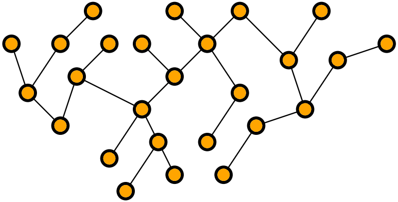

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Seweryn 2020

If cover graph of **P** has treewidth $\leqslant 2$ then dim(**P**) $\leqslant 12$

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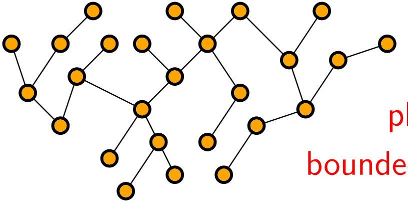

planar?

bounded treewidth?

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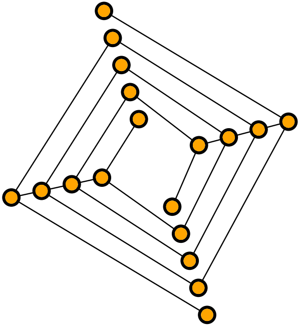
Kelly's example

Kelly 1981



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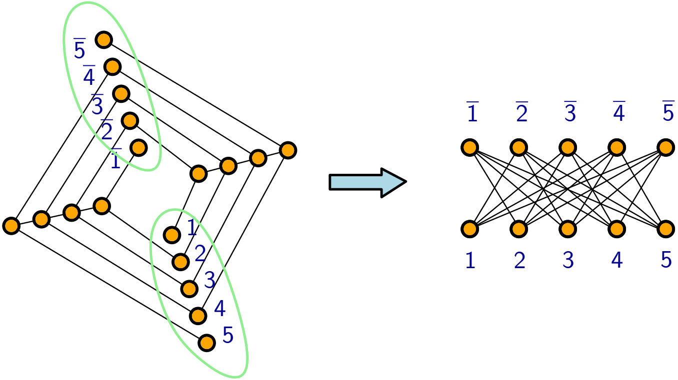
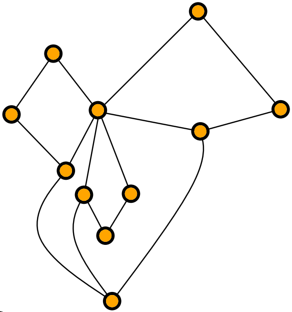


planar posets with arbitrarily large dimension
cover graphs have treewidth 3

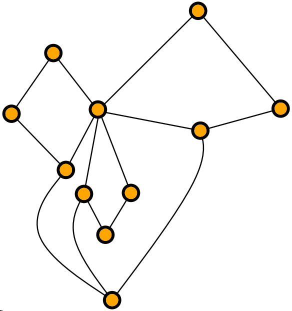
What now?



1. Are planar posets dim-bounded?
Posets with planar cover graphs?

Conjecture (Trotter, 1980s): Yes and yes

What now?



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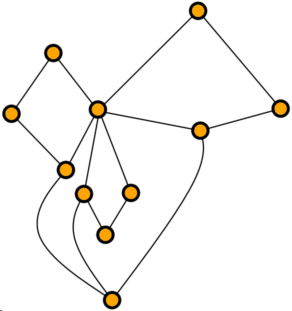
2. Are posets with cover graphs of bounded treewidth dim-bounded?

What now?



1. Are planar posets dim-bounded?
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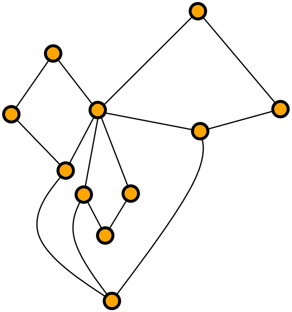
2. Are posets with cover graphs of bounded treewidth dim-bounded?

3. What properties of cover graphs imply bounded dimension?

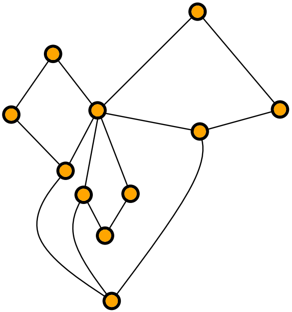
Posets with planar cover graphs



Blake, Hodor, Micek, Seweryn, Trotter 2023+

Posets with planar cover graphs are dim-bounded

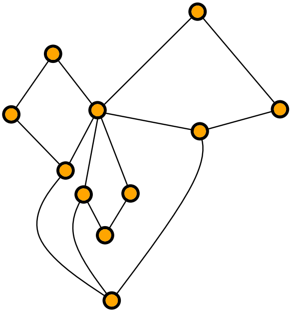
Posets with planar cover graphs



Blake, Hodor, Micek, Seweryn, Trotter 2023+
Posets with planar cover graphs are dim-bounded

Blake, Hodor, Micek, Seweryn, Trotter 2023+
If $\mathbf{P}$ has a planar cover graph then $\dim(\mathbf{P}) \leqslant 2^{O(\mathrm{se}(P))}$

Posets with planar cover graphs



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Blake, Hodor, Micek, Seweryn, Trotter 2023+
If $\mathbf{P}$ has a planar diagram then $\dim(\mathbf{P}) \leqslant 128\,\mathrm{se}(\mathbf{P}) + O(1)$

Posets with cover graphs of bounded treewidth

J., Micek, Mi. Pilipczuk, Walczak 2023

Posets with cover graphs of bounded treewidth are dim-bounded

Posets with cover graphs of bounded treewidth

J., Micek, Mi. Pilipczuk, Walczak 2023
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J., Micek, Mi. Pilipczuk, Walczak 2023
Posets with bounded cliquewidth are dim-bounded

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J., Micek, Mi. Pilipczuk, Walczak 2023
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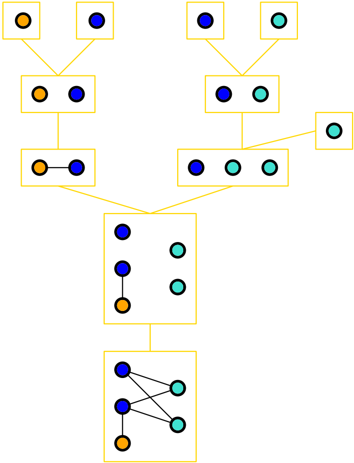
Posets with bounded cliquewidth are dim-bounded

New tool: Colcombet's factorization theorem

Graph cliquewidth



Build graph G using 4 operations:

- create new vertex v with label i
- disjoint union of two labeled graphs
- put all edges between vertices labeled i and vertices labeled j $(i \neq j)$
- rename label i to j

Cliquewidth: min. number of labels needed

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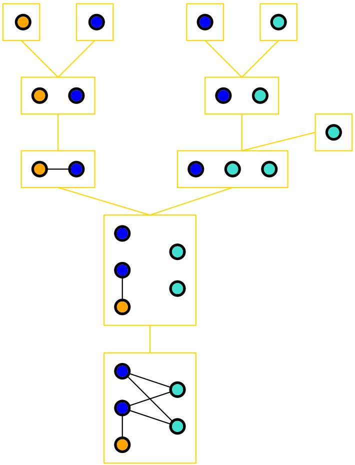
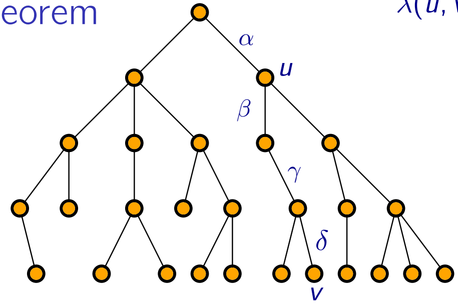


Cliquewidth: min. number of labels needed

Poset cliquewidth: Same for posets

Colcombet's Theorem

$\lambda(u, v) = \beta \cdot \gamma \cdot \delta$



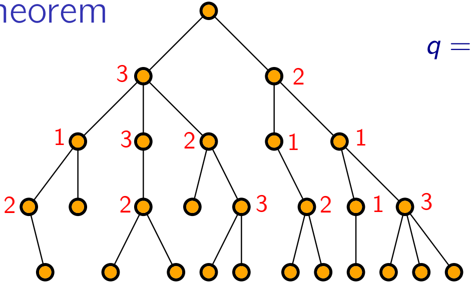
Rooted tree T

Edges labeled with elements of a finite semigroup $(\Lambda, \cdot)$

$u <_T v$ means "u strict ancestor of v"

For $u <_T v$, set $\lambda(u, v) :=$ product of labels on u-to-v path

Colcombet's Theorem



$q = 3$

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Split of order q: each inner node receives some number in $\{1, \dots, q\}$

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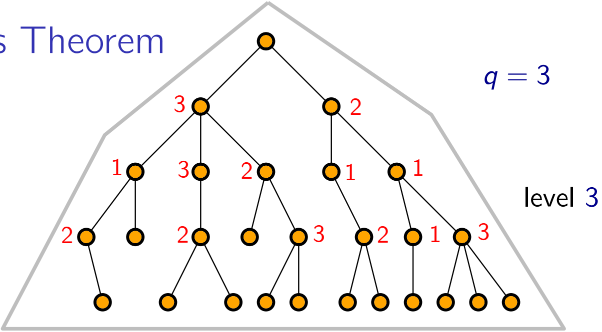

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Hierarchical factorization of T

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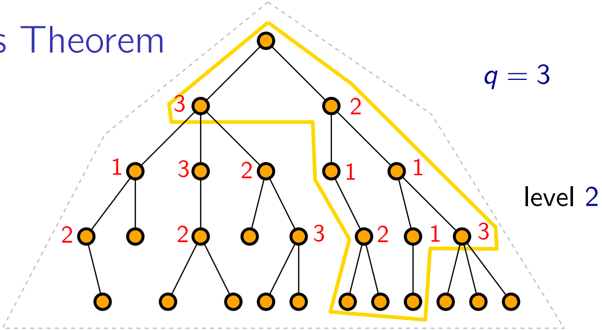

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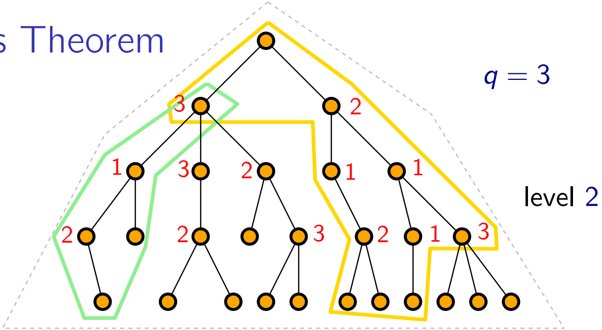

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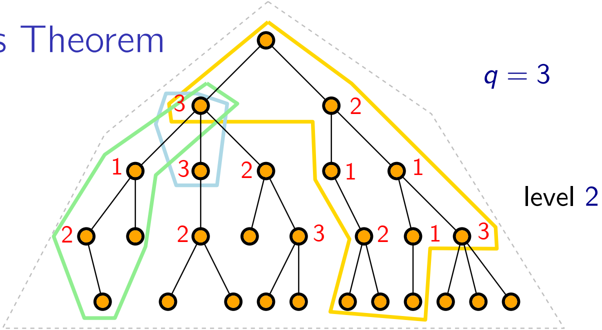

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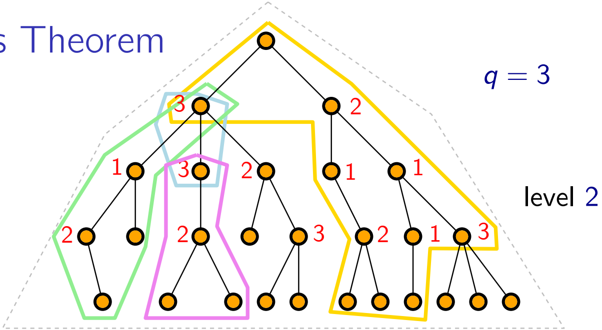

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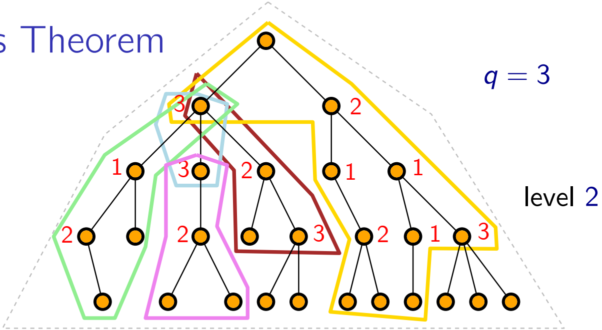

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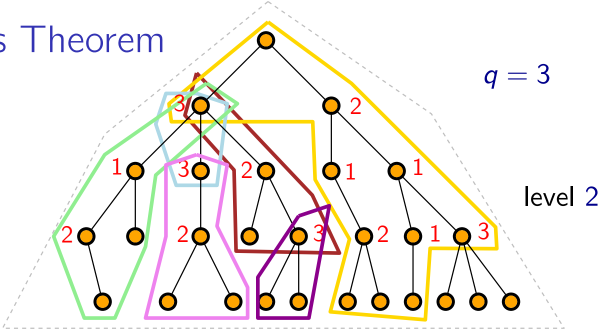
For $u <_T v$, set $\lambda(u, v) :=$ product of labels on u-to-v path

Split of order q: each inner node receives some number in $\{1, \ldots, q\}$

Hierarchical factorization of T

Colcombet's Theorem



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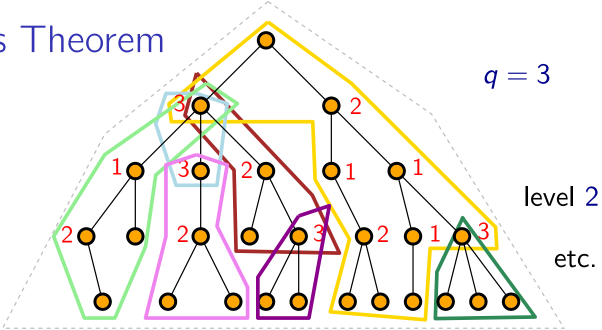

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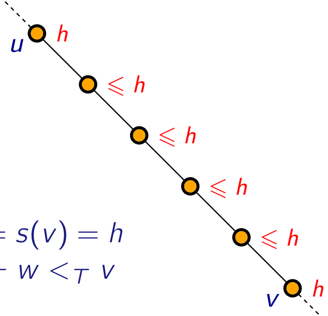

$u <_T v$ are h-**neighbors** if $s(u) = s(v) = h$ and $s(w) \leqslant h$ for all w with $u <_T w <_T v$

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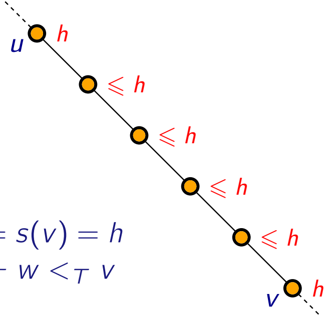

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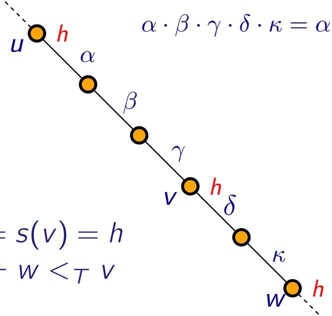

$\alpha \cdot \beta \cdot \gamma \cdot \delta \cdot \kappa = \alpha \cdot \beta \cdot \gamma$

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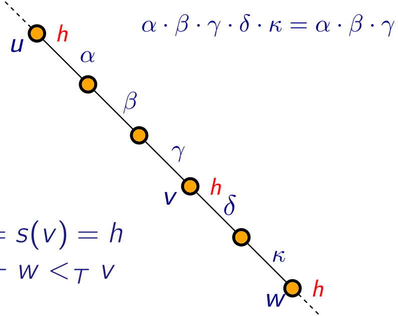

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Colcombet 2007
There exists a split of T of order $|\Lambda|$ that is forward Ramseyan

Inspiration:

Bonamy and Mi. Pilipczuk 2020
Graphs of bounded cliquewidth are polynomially χ-bounded

Nešetřil, Ossona de Mendez, Mi. Pilipczuk, Rabinovich, Siebertz 2021
Graphs with bounded cliquewidth excluding some half-graph as a semi-induced subgraph are linearly χ-bounded

J., Micek, Mi. Pilipczuk, Walczak 2023
Posets with bounded cliquewidth are dim-bounded

Absolute bounds on dimension

Trotter and Moore 1977

If cover graph of $\mathbf{P}$ is a forest then $\dim(\mathbf{P}) \leqslant 3$

Seweryn 2020

If cover graph of $\mathbf{P}$ has treewidth $\leqslant 2$ then $\dim(\mathbf{P}) \leqslant 12$

Absolute bounds on dimension

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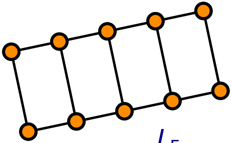

L_5

Huynh, J., Micek, Seweryn, Wollan 2022

If cover graph of **P** has no L_k minor then $\dim(\mathbf{P}) \leqslant f(k)$

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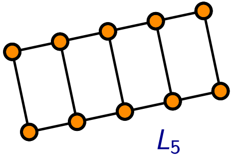

Huynh, J., Micek, Seweryn, Wollan 2022

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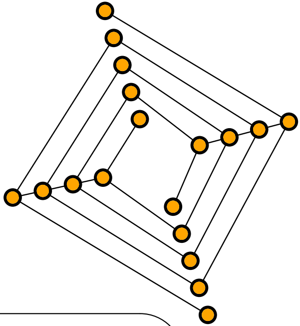
Can we characterize minor-closed graph classes $\mathcal{G}$ s.t. posets with cover graphs in $\mathcal{G}$ have bounded dimension?

Kelly's examples again



J., Micek, Mi. Pilipczuk, Walczak 2023

Fix a minor-closed graph class $\mathcal{G}$.

Then posets with cover graphs in $\mathcal{G}$ have bounded dimension $\Leftrightarrow$ $\mathcal{G}$ excludes the cover graph of some Kelly example

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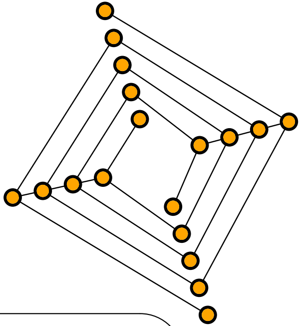

J., Micek, Mi. Pilipczuk, Walczak 2023
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J., Micek, Mi. Pilipczuk, Walczak 2023
For fixed t, k, every poset with large enough dimension and whose cover graph has treewidth $\leqslant t$ contains the Kelly example of order k as a subposet

Boolean Dimension

Boolean realizer of $\mathbf{P}$: sequence of k linear orders $L_1, \ldots, L_k$ on elements of P and a k-ary Boolean function ϕ s.t.

$$x \leqslant y \text{ in } P \quad \Leftrightarrow \quad \phi\big((x \leqslant y \text{ in } L_1), \ldots, (x \leqslant y \text{ in } L_k)\big) = 1$$

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$\mathrm{bdim}(\mathbf{P}) \leqslant \dim(\mathbf{P})$

$\mathrm{bdim}(S_k) = 4$

Felsner, Mészáros, Micek 2020

Posets with cover graphs of bounded treewidth have bounded Boolean dimension

J., Micek, Mi. Pilipczuk, Walczak 2023

Posets with bounded cliquewidth have bounded Boolean dimension

Open problems

Conjecture: Fix a proper minor-closed graph class $\mathcal{G}$. Then posets with cover graphs in $\mathcal{G}$ are dim-bounded.

True for planar graphs and graphs of bounded treewidth

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Conjecture (Nešetřil and Pudlák, 1989): Posets with planar cover graphs have bounded Boolean dimension

INNOVATIONS IN GRAPH THEORY