Angry theorems and decompositions of 3-connected graphs

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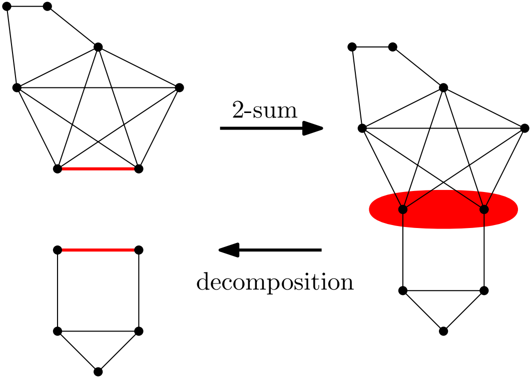
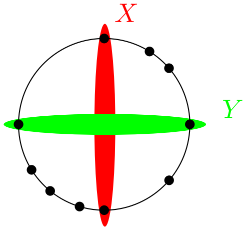
Joint work with Jan Kurkofka

Decomposing 2-connected graphs





X crosses Y if X separates Y.

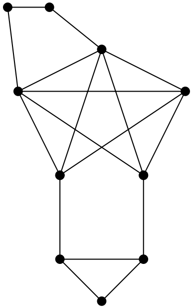
Tutte's Angry theorem (1961)

A 2-connected graph all whose 2-separatos are crossed is a cycle or 3-connected.

Decomposing 2-connected graphs

Tutte's 2-separator theorem (1961)

Every 2-connected graph has a canonical tree-decomposition of adhesion 2 all whose torsos are 3-connected or cycles.



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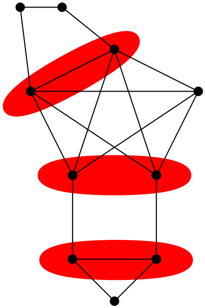


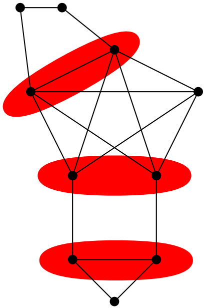
Applications:

- Group Theory
- Computer Science
  (FPT algorithms, cobs and robbers games,..)
- Structural Graph Theory

A simple application in Geometric Group Theory

Fact

A 2-connected vertex-transitive graph is 3-connected or a cycle.



Other angry theorems

- Cunningham's angry theorem for 1-joins;
- recent works on 1-separations in digraphs by Bowler, Gut, Hatzel, Kawarabayashi, Muzi, Reich;

Local Tutte Theorem (C 2023)

Given $r \in \mathbb{N}$, an r-locally 2-connected graph all of whose r-local 2-separators are crossed is a cycle of length at most r or r-locally 3-connected.

- More???

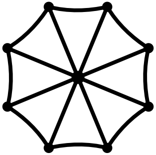
today

Part I: there is no angry theorem for 3-separators (1961-2022)

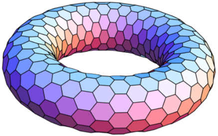
Part II: an angry theorem for separators of size 3 (2023)

Part III: Outlook and Applications

Wheels are 3-angry



Internally 4-connected graphs are 3-angry

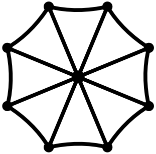


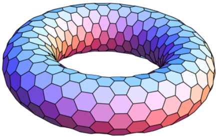
A 3-connected graph is internally 4-connected if all its 3-separators X leave only two components and one of them is a singleton, and X is an anti-clique.

An angry theorem for 3-separators?

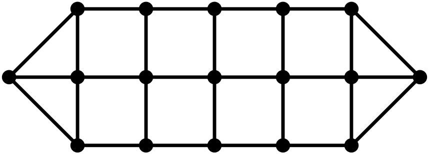
Conjecture (false)

A 3-connected graph in which every 3-separator is crossed is a wheel or internally 4-connected.

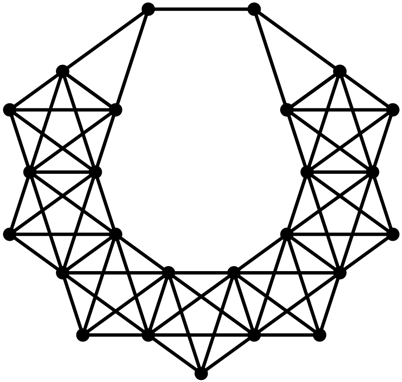




Counterexample 1



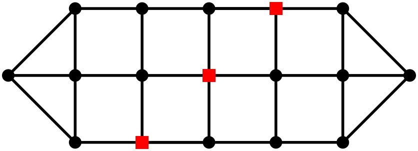
Counterexample 2



Part II: an angry theorem for separators of size 3

Our perspective

A tri-separator is a separator consisting of three vertices or edges, where vertices are replaced by edges if possible, roughly speaking.



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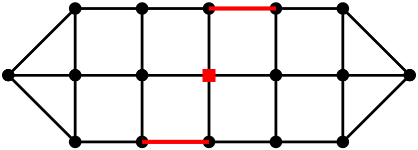

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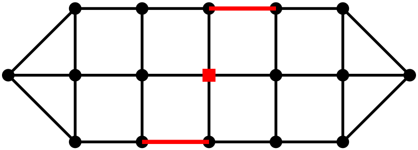
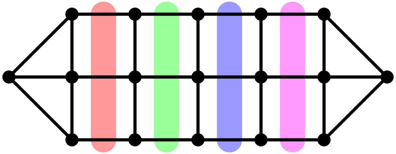
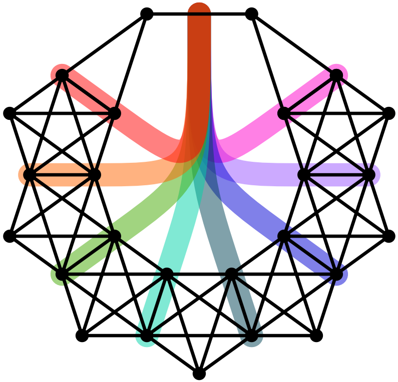


Intuition behind this

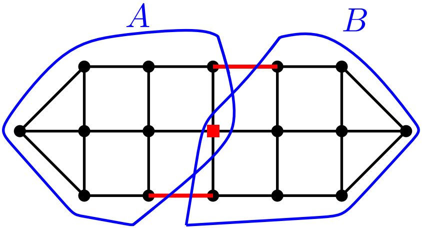
- same notion of separability BUT
- smoother notion of crossing.



~~Counter~~ example 2



Formal definition



A mixed separation is a pair (A, B) such that $V(G) = A \cup B$ and $A \setminus B$ and $B \setminus A$ are nonempty.
Its separator is $A \cap B$ together with $E(A \setminus B, B \setminus A)$.

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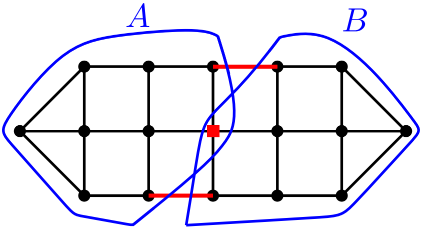

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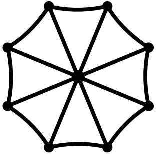
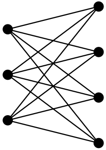
A tri-separation is a mixed 3-separation such that every vertex in A or B has two neighbours in A or B, respectively. A tri-separator is the separator of a tri-separation.
A tri-separation (A, B) is trivial if A or B consists of a single vertex.

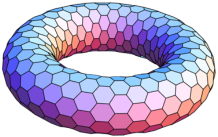
Main theorem

Angry tri-separator theorem (C, Kurkofka 2023)

A 3-connected graph in which every nontrivial tri-separator is crossed is a wheel, $K_{3,m}$, or internally 4-connected.

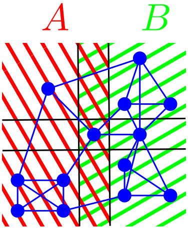






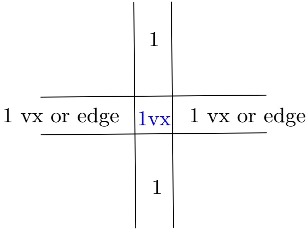
Note:
G internally 4-connected $\leftrightarrow$ all tri-separators are trivial and $G \neq K_{3,3}$.

Corner diagrams



Proof: step 1

Extend the theory from crossing 3-separators to tri-separators.



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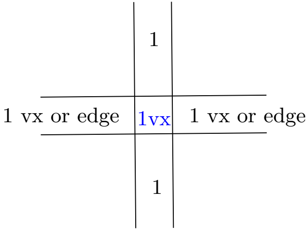

So: $\exists v \in V(G)$ so that $G - v$ has two crossing 2-separators (or a few special cases).

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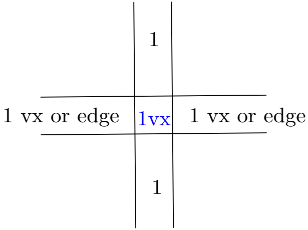

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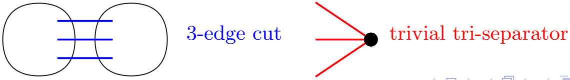
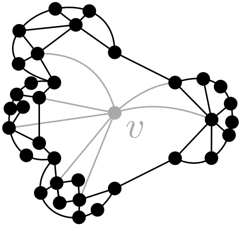
Example

3-edge cuts are always tri-separators and can only be crossed if they are trivial.



Proof: step 2

Apply Tutte's 2-separator theorem to $G - v$.



$G - v$ is obtained from 2-connected graphs and single edges by 2-summing them at a cycle.

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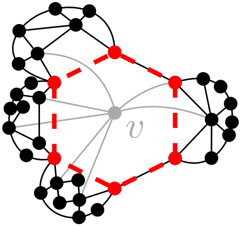

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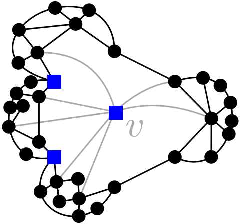
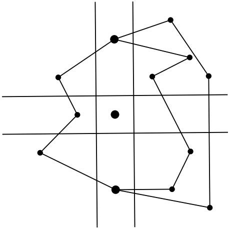


Candidate for a tri-separator that is not crossed.

Proof: step 3

Idea: characterise when a tri-separator is not crossed via a connectivity property.



Example

A tri-separator consisting of three vertices is not crossed if any two of its vertices are adjacent or joined by three internally disjoint paths avoiding the third vertex.

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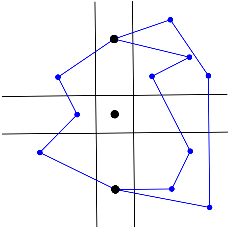

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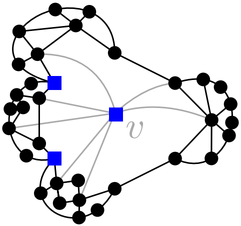
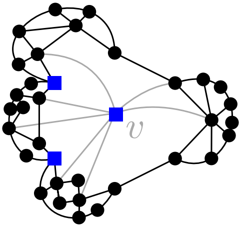
Proof: final step, in a special case



Claim: X is not crossed.



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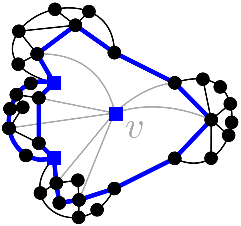

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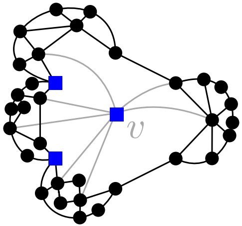

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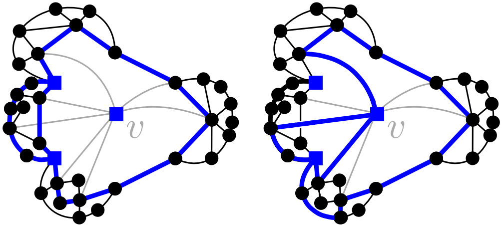

Summary of the proof

Let G be an angry graph that is not internally 4-connected.

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  If G is not a wheel and not $K_{3,m}$, then this decomposition is 'nontrivial';
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Decomposition theorem for 3-connected graphs

A tri-separation is totally nested if it is not crossed by a tri-separation.

Tri-separator theorem (C, Kurkofka 2023, vague version)

If we cut at all totally nested nontrivial tri-separations simultaneously all pieces are essentially angry graphs.

Tri-separator theorem (C, Kurkofka 2023, version from paper)

Let G be a 3-connected graph and let N denote its set of totally-nested nontrivial tri-separations. Each torso τ of N is a minor of G and satisfies one of the following:

- τ is quasi 4-connected;
- τ is a wheel;
- τ is a thickened $K_{3,m}$ or $G = K_{3,m}$ with $m \geq 0$.

Application 1

λ-connectivity augmentation problem

Input:

numbers n, k, graph G on n vertices and set F of edges outside G.

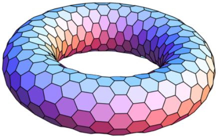
Decide:

is there $X \subseteq F$ with $|X| \leq k$ such that $G + X$ is λ-connected?

Theorem (C, Sridharan 2023$^+$)

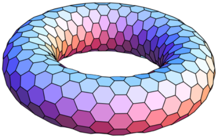
For every $\lambda \leq 4$, λ-connectivity augmentation can be decided in time $f(k) \cdot n^{\mathcal{O}(1)}$, for some function f.

Application 2



Second example: replace vertices by triangles!

Application 2



Second example: replace vertices by triangles!

A graph G is almost 4-connected if it is 3-connected, and the removal of every 3-separator leaves exactly two components and one component with at most three vertices.

Corollary

A 3-connected vertex-transitive graph is a complete graph, $K_{3,3}$ or almost 4-connected graph.

(stronger version in the paper)

Question

Is there a decomposition theorem that works for all k?

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Yes: Robertson Seymour Graph Minor structure theorem

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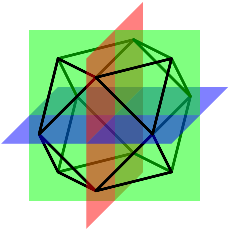
Structure theorem $\rightarrow$ existence results via a coarse topological structure
Tutte's theorem $\rightarrow$ excluded minors for series-parallel graphs
Tri-separator theorem $\rightarrow$ excluded minors for planar graphs

Question

Is there an angry theorem for 4-connected graphs?



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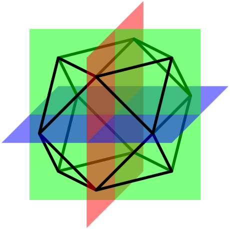

Question

Is there an angry theorem for 3-connected matroids?

Question

Is there an angry theorem for 3-connected graphs G via bipartitions of G of rank at most two?