2^{nd} TUTORIAL ON RANDOMIZED ALGORITHMS

Walking randomly on graphs

1. *Random walks warm-up.* Here's a simple example of a random walk on the discrete line: Let $n \in \mathbb{N}$ be an even number. Let us the following problem we start with $X_0 = n/2$ and do the following process:

- if $X_i \in \{0, n\}$ we stop, and otherwise,
- we set $X_{i+1} = X_i + \delta$, where δ is picked uniformly at random from $\{-1, 1\}$

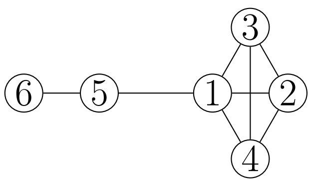
1. Is this a Markov chain? If so can you write its matrix?
2. How to compute the expected number of steps until stopping? You can skip the computation if you know how to do it.

2. *Coupon collector.* We would like to collect all n kinds of coupons. Coupons are sold in packages which all look the same. Thus when we buy an coupon, we buy one of n kinds uniformly at random.

1. What is the expected number of coupons we need to buy to get all kinds?
2. How many coupons do we need to buy to have probability at least $1 - q$ of collecting all kinds?
3. What is the Markov chain? Is this similar to a random walk on some graph?

3. *Walking on a lollipop.* A lollipop is an undirected graph on n vertices consisting of a complete graph on t vertices and a path on $s = n - t$ additional vertices, connected by an edge from one endpoint to one of the vertices of the complete graph. Let u be some other vertex of the complete graph (i.e., some vertex of degree $t-1$) and v be the other endpoint of the path (i.e., the single vertex of degree 1). Consider the random walk on this graph.

1. Compute the hitting time h_{uv}, i.e., the expected time of reaching v from u, as well as h_{vu}. It is sufficient to do it *asymptotically.*
2. Determine also the values of t for which the hitting times are maximized or minimized.



4. *Oriented graphs.*

1. Find a family of oriented graphs with as large hitting time as possible.
2. What if we additionally require that they have constant in-degree and constant out-degree?

5. *Random Tom and Jerry on a cycle.* A cat and a mouse each independently take a random walk on a cycle of length $2n+1$. (In each step, they both take a random step, i.e., both random walks are synchronized.) The game ends (the cat eats the mouse) if they land at the same vertex at the same time. (If they traverse the same edge in the opposite directions, nothing happens.) Find the expected number of steps of the game if the cat and the mouse start (a) at adjacent nodes and (b) at distance n (i.e., at the largest possible distance).