1. We are coloring faces of the regular tetrahedron using k colors. We consider two colorings to be the same if they differ only by a rotation of the tetrahedron (if they differ only by a reflection but not only by a rotation, we consider them to be different). How many different colorings are there?

2. You are building a necklace of n beads of k different colors joined in a cycle. Two necklaces are considered to be the same if they differ only by a rotation of the cycle. Show that the number of such necklaces is equal to
$$\frac{1}{n}\sum_{b=1}^{n} k^{\gcd(b,n)}.$$

3. For a positive integer m, let $\phi(m)$ denote the number of integers $i \in \{1,\ldots,m\}$ such that $\gcd(i,m) = 1$. Show that for all positive integers n and g, the number of integers $b \in \{1,\ldots,n\}$ such that $\gcd(b,n) = g$ is
$$\begin{cases} 0 & \text{if } g \text{ does not divide } n \\ \phi(n/g) & \text{if } g \text{ divides } n. \end{cases}$$
Use this to simplify the answer to the previous exercise.

4. Show that the chromatic polynomial of the n-cycle is equal to $(x-1)^n + (-1)^n(x-1)$. Hint: Use the deletion-contraction formula and induction on n.

5. Determine the number of proper k-colorings of the 6-cycle, where two colorings are considered to be the same if they differ only by an automorphism of the cycle (i.e., a rotation or a reflection).

6. Compute the cycle index $Z_4(x_1,x_2,x_3,x_4)$ of the group Sym_4 of all permutations of the set $\{1,2,3,4\}$.

7. For each n, let t_n denote the number of rooted trees where each non-leaf node has exactly four children, and the order of children *does not* matter. Let $T(x) = \sum_{n=0}^{\infty}$ be the corresponding generating function. Show that
$$T(x) = x\cdot\Big(1 + Z_4(T(x),T(x^2),T(x^3),T(x^4))\Big).$$