Chapter 5

Connectivity and separation

In this chapter, we develop notions analogous to graph connectivity and cuts.

5.1 Connectivity, separation and unions

We have already seen that the matroid $\mathcal{M}(G)$ associated with a graph G remains the same when permuting blocks of G. Recall that a graph G is *k-connected* if it has at least $k+1$ vertices and it stays connected after removing any at most $k-1$ vertices and a *block* of a graph G is an inclusion-wise maximal 2-connected subgraph of G or an edge contained in no 2-connected subgraph. Hence, there is no chance to determine whether G is connected from its graphic matroid and it seems natural to continue with G being 2-connected. Before we further explore this, let us state an auxiliary proposition on 2-connected graphs.

Proposition 5.1. *Let G be a loopless graph. The graph G is 2-connected if and only if every two edges e and f of G lie on a common cycle C of G.*

Proposition 5.1 suggest a possible way of generalizing the notion of connectivity to matroids. Let $\mathcal{M}$ be a matroid with a ground set E. We define a binary relation $\gamma_{\mathcal{M}}$ on E as follows: $(e,f) \in \gamma_{\mathcal{M}}$ if $e = f$ or there exists a circuit C in $\mathcal{M}$ such that $\{e,f\} \subseteq C$. We will show that the relation $\gamma_{\mathcal{M}}$ is an equivalence relation. In order to do so, we will need the following lemma on circuits of matroid which is known as the *strong circuit elimination axiom*.

Lemma 5.2. *Let $\mathcal{C}$ be a family of circuits of a matroid $\mathcal{M}$. Then, the following holds:*

(C3)' *if $C_1, C_2 \in \mathcal{C}$, $e \in C_1 \cap C_2$ and $f \in C_1 \setminus C_2$ then there is $C \in \mathcal{C}$ such that $f \in C \subseteq (C_1 \cup C_2) - e$.*

Proof. Assume that (C3)' fails and choose two C_1 and C_2 violating (C3)' such that the size $|C_1 \cup C_2|$ is as small as possible. By the property (C3) from Lemma 1.1, there exists a circuit $C_3 \subset (C_1 \cup C_2) - e$ but $f \notin C_3$. Clearly $e \in C_2 \setminus C_3$ and since $C_3 \not\subseteq C_1$ there is $g \in C_2 \cap C_3$. Moreover, $C_2 \cup C_3 \subseteq (C_1 \cup C_2) - f$ and thus $|C_2 \cup C_3| < |C_1 \cup C_2|$. In particular, (C3)' holds for C_2, C_3 and $g \in C_2 \cap C_3$ and $e \in C_2 \setminus C_3$, i.e., there exists a circuit C_4 with $e \in C_4 \subseteq (C_2 \cup C_3) - g$. Again, $C_1 \cup C_4 \subseteq (C_1 \cup C_2) - g$ and $|C_1 \cup C_4| < |C_1 \cup C_2|$. Hence, we can apply (C3)' for C_1 and C_4 with $e \in C_1 \cap C_4$ and $f \in C_1 - C_4$ and find a circuit C such that $f \in C \subseteq (C_1 \cup C_4) - e \subseteq (C_1 \cup C_2) - e$ which is a contradiction. □

We are now ready to show that the relation $\gamma_{\mathcal{M}}$ is an equivalence relation.

Lemma 5.3. *The relation $\gamma_{\mathcal{M}}$ is an equivalence relation for every matroid $\mathcal{M}$.*

Proof. The relation $\gamma_{\mathcal{M}}$ is clearly reflexive and symmetric. In order to prove its transitivity, let $(e, f) \in \gamma$ and $(f, g) \in \gamma$ where e, f, g are distinct elements of E. By the definition of $\gamma_{\mathcal{M}}$, there exist circuits C_1 and C_2 such that $e \in C_1$, $g \in C_2$ and $C_1 \cap C_2 \neq \emptyset$. Choose such circuits C_1 and C_2 with the size $|C_1 \cup C_2|$ as small as possible. Assume that there is no circuit of $\mathcal{M}$ containing both e and g. Let $h \in C_1 \cap C_2$, by Lemma 5.2, there exists a circuit C_3 such that $e \in C_3 \subseteq (C_1 \cup C_2) - h$. Moreover, $g \notin C_3$.

Since C_3 cannot be a subset of C_1, there exists $i \in C_2 - C_1$ and $i \in C_3$. We now use Lemma 5.2 for circuits C_2 and C_3 to obtain a circuit C_4 such that $g \in C_4 \subseteq (C_2 \cup C_3) - i$. Since C_4 cannot be a subset of C_2, it holds that $C_4 \cap (C_3 \setminus C_2) \neq \emptyset$. Since $C_3 \setminus C_2 \subseteq C_1$, we also get $C_1 \cap C_4 \neq \emptyset$. But $C_1 \cup C_4 \subseteq (C_1 \cup C_2) - i$ and $|C_1 \cup C_4| < |C_1 \cup C_2|$. This contradicts the choice of C_1 and C_2. □

The equivalence classes of $\gamma_{\mathcal{M}}$ are called *connectivity components* of $\mathcal{M}$. A matroid $\mathcal{M}$ is *connected* if the equivalence relation $\gamma_{\mathcal{M}}$ has a single equivalence class. The definition of the relation of $\gamma_{\mathcal{M}}$ immediately implies the following:

Proposition 5.4. *A matroid $\mathcal{M}$ is connected if and only if every two elements of $\mathcal{M}$ lie in a common circuit of $\mathcal{M}$.*

Propositions 5.1 and 5.4 combine to the following:

Proposition 5.5. *Let G be a loopless graph with at least 3 vertices. The graph G is 2-connected if and only if the matroid $\mathcal{M}(G)$ is connected.*

Proof. If G is 2-connected, then any two edges are contained in a common cycle of G and thus in a common circuit of $\mathcal{M}(G)$. Similarly, if $\mathcal{M}(G)$ is connected, then any two edges of G are contained in a common circuit of $\mathcal{M}(G)$ and thus a common cycle of G. The statement easily follows. □

A subset X of the ground set $E(\mathcal{M})$ of a matroid $\mathcal{M}$ is called *a separator* if it is a union of connected components of $\mathcal{M}$. The next lemma immediately follows from the definition of connectivity components.

Lemma 5.6. *Let $\mathcal{M}$ be a matroid with a ground set E. A subset $X \subseteq E$ is a separator if and only if every circuit C of $\mathcal{M}$ is contained entirely in X or in $E \setminus X$.*

Separators of matroids can also be characterized using the rank function.

Proposition 5.7. *Let $\mathcal{M}$ be a matroid with a ground set E. A subset $X \subseteq E$ is a separator if and only if $r(X) + r(E \setminus X) = r(\mathcal{M})$.*

Proof. The submodularity of the rank function of a matroid implies that $r(X) + r(E \setminus X) \geq r(\mathcal{M})$ for any $X \subseteq E$.

Let B_X and $B_{E\setminus X}$ be the maximal independent sets of X and $E \setminus X$, respectively, and $B = B_X \cup B_{E\setminus X}$. If X is a separator, then B is independent in $\mathcal{M}$ and $r(X) + r(E \setminus X) = |B_X| + |B_{E\setminus X}| = |B| \leq r(\mathcal{M})$ and the statement follows.

On the other hand, if X is not a separator, there exists a circuit C intersecting both X and $E \setminus X$. Consider inclusion-wise maximal subsets B'_X and $B'_{E\setminus X}$ of X and $E\setminus X$, respectively, containing (independent) sets $C \cap X$ and $C \cap (E \setminus X)$. The set $B' = B'_X \cup B'_{E\setminus X}$ contains C and thus it is dependent. Moreover, since B'_X and $B'_{E\setminus X}$ are inclusion-wise maximal and $r(B') = r(\mathcal{M})$. Hence, $r(X) + r(E \setminus X) = |B'_X| + |B'_{E\setminus X}| = |B'| > r(B') = r(\mathcal{M})$ and the equality does not hold. □

The rank function of a matroid behaves independently on its separators as we state in the next lemma.

Lemma 5.8. *Let $\mathcal{M}$ be a matroid with a ground set E and X a separator of $\mathcal{M}$. For every subsets F of the ground set E, the following holds:*

$$r(F) = r(F \cap X) + r(F \setminus X)\,.$$

Proof. Choose inclusion-wise maximal independent subsets F_X and $F_{E\setminus X}$ of $F \cap X$ and $F \setminus X$, respectively. Clearly, $r(F \cap X) = |F \cap X|$ and $r(F \setminus X) = |F \setminus X|$. If $r(F) < r(F \cap X) + r(F \setminus X)$, then the set $F_X \cup F_{E\setminus X}$ is not independent and thus it contains a circuit C. Clearly, C has a non-empty intersection with both F_X and $F_{E\setminus X}$. The existence of C contradicts the fact that X is a separator. On the other hand, the submodularity of the rank function implies that $r(F) \leq r(F \cap X) + r(F \setminus X)$ and thus the equality must hold. □

We now characterize separators of matroids using the contraction and deletion operations. Before we do so, we need the following lemma.

Lemma 5.9. *Let $\mathcal{M}$ be a matroid and X a subset of its ground set. The matroids $\mathcal{M} \setminus X$ and $\mathcal{M}/X$ are the same if and only if $r(X) + r(E \setminus X) = r(\mathcal{M})$.*

Proof. Assume that $\mathcal{M} \setminus X = \mathcal{M}/X$ and let B be a base of $\mathcal{M} \setminus X = \mathcal{M}/X$. If B_X is a base of $\mathcal{M}|X$, then $B \cup B_X$ is a base of $\mathcal{M}$ and $r(\mathcal{M}) = |B| + |B_X| = r(\mathcal{M} \setminus X) + r(\mathcal{M}|X) = r(E\setminus) + r(X)$.

In the other direction, assume that the equality hold. First, observe that $\mathcal{I}(\mathcal{M}/X) \subseteq \mathcal{I}(\mathcal{M} \setminus X)$. In order to prove the opposite inclusion, consider $I \in \mathcal{I}(\mathcal{M} \setminus X)$ and a base B of $\mathcal{M} \setminus X$ containing I. Clearly, B can be completed to a base $B \cup B'$ of $\mathcal{M}$. The equality $r(\mathcal{M}) = |B| + |B'| = r(E \setminus X) + |B'|$ implies that $|B'| = r(X)$ and B' is a base of $\mathcal{M}|X$. Hence, B is a base of $\mathcal{M}/X$ and $I \in \mathcal{I}(\mathcal{M}/X)$. We conclude that $\mathcal{I}(\mathcal{M}/X) = \mathcal{I}(\mathcal{M} \setminus X)$. □

Proposition 5.7 and Lemma 5.9 yield a characterization of matroid separators using the contraction and deletion operations.

Lemma 5.10. *Let $\mathcal{M}$ be a matroid and X a subset of its ground set. The subset X is a separator of $\mathcal{M}$ if and only if the matroids $\mathcal{M} \setminus X$ and $\mathcal{M}/X$ are the same.*

Propositions 2.4 and 5.7 combine together to the following:

Proposition 5.11. *Let $\mathcal{M}$ be a matroid and X a subset of its ground set. X is a separator if and only if $r(X) + r^*(X) - |X| = 0$.*

Since the formula from Proposition 5.11 is self-dual, we obtain the following.

Lemma 5.12. *A matroid $\mathcal{M}$ is connected if and only if its dual $\mathcal{M}^*$ is connected.*

We now focus on matroids that are not connected. We give the description how to "build" them from connected pieces. Proposition 5.7 and Lemma 5.8 yield the following.

Proposition 5.13. *Let $\mathcal{M}$ be a matroid with a ground set E. The matroid $\mathcal{M}$ is not connected if and only if there exists a proper subset X of E such that*

$$\mathcal{I}(\mathcal{M}) = \{I_1 \cup I_2 \mid I_1 \in \mathcal{I}(\mathcal{M}|X), I_2 \in \mathcal{I}(\mathcal{M} \setminus X)\}.$$

Propositions 1.21 and 5.13 allows to break down a matroid into its connected pieces using the notion of the union of two matroids.

Theorem 5.14. *Let $\mathcal{M}$ be a matroid with connectivity components X_1, X_2, …, X_k. The matroid $\mathcal{M}$ is equal the matroid union*

$$\mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \ldots \oplus \mathcal{M}_k$$

where $\mathcal{M}_i = \mathcal{M}|X_i$. Moreover, if $\mathcal{M}$ is isomorphic to

$$\mathcal{N}_1 \oplus \mathcal{N}_2 \oplus \ldots \oplus \mathcal{N}_l$$

where all matroids $\mathcal{N}_i$, $i = 1 \ldots, l$, are connected, then $k = l$ and there exists a permutation σ of $\{1, 2, \ldots, k\}$ such that the matroids $\mathcal{N}_i$ and $\mathcal{M}_{\sigma(i)}$ are isomorphic.

Proof. We first prove that $\mathcal{M}$ is equal to the union of $\mathcal{M}_i$, $i = 1, \ldots, k$. Since X_k is a separator, $\mathcal{M} = (\mathcal{M} \setminus X_k) \oplus (\mathcal{M}|X_k)$ by Proposition 1.21 and Lemma 5.8. By the induction, it holds that $\mathcal{M} \setminus X_k = \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \ldots \oplus \mathcal{M}_{k-1}$.

Assume now that $\mathcal{M}$ is isomorphic to the matroid union $\mathcal{N}_1 \oplus \mathcal{N}_2 \oplus \ldots \oplus \mathcal{N}_l$ as described in the statement of the theorem. Clearly, the ground set of each of the matroids $\mathcal{N}_i$, $i = 1, \ldots, l$, is an equivalence class of $\gamma_{\mathcal{M}}$. The correspondence between the equivalence classes yields the permutation σ and the isomorphism between $\mathcal{N}_i$ and $\mathcal{M}_{\sigma(i)}$ is then obtained as a restriction of the isomorphism between $\mathcal{N}_1 \oplus \mathcal{N}_2 \oplus \ldots \oplus \mathcal{N}_l$ and the matroid $\mathcal{M}$. □

We finish this section by a proposition asserting that several classes of matroids being closed under unions.

Proposition 5.15. *The classes of $\mathbb{F}$-representable, graphic and cographic matroids are closed under the matroid union.*

Proof. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two matroids represented over a field $\mathbb{F}$. Let A_i be a representation of $\mathcal{M}_i$, $i = 1, 2$. It is easy to verify that the matrix $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ is a representation of $\mathcal{M}_1 \oplus \mathcal{M}_2$ over $\mathbb{F}$. Hence, the class of $\mathbb{F}$-representable matroids is closed under matroid unions.

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two graphic matroids corresponding to graphs G_1 and G_2, respectively. The matroid $\mathcal{M}_1 \oplus \mathcal{M}_2$ is then a graphic matroid corresponding to the graph obtained by a disjoint union of G_1 and G_2. This shows that the class of graphic matroids is closed under matroid unions. By Proposition 2.7, the union of two cographic matroids is also cographic. □

Proposition 5.15 also implies that the class of regular matroids, which is defined in Section 6.3, is closed under the matroid union.

5.2 Tutte, essential and cyclic connectivity

In this section, we develop analogoues of several notions of connectivity for matroids and show their counterparts for graphs. Let us start with the following proposition as a motivation for our definitions.

Proposition 5.16. *Let G be a connected graph and let (X, Y) be a partition of its edge set. Let G_X and G_Y be the subgraphs of G formed by the edges contained in X and Y, respectively, and the vertices incident with these edges. The following holds:*

$$r_{\mathcal{M}(G)}(X) + r_{\mathcal{M}(G)}(Y) - r_{\mathcal{M}(G)}(\mathcal{M}(G)) \leq |V(G_X) \cap V(G_Y)| - 1\,.$$

Moreover, if both the subgraphs G_X and G_Y are connected, then the inequality is an equality.

Proof. Let n_X and n_Y be the number of vertices of G_X and G_Y, respectively, and let $c(G_X)$ and $c(G_Y)$ be the numbers of their components. Since $r_{\mathcal{M}(G)}(X) = n_X - c_X$ and $r_{\mathcal{M}(G)}(Y) = n_Y - r_Y$, we obtain the following:

$$\begin{aligned} & r_{\mathcal{M}(G)}(X) + r_{\mathcal{M}(G)}(Y) - r_{\mathcal{M}(G)}(X \cup Y) + 1 \\ &= n_X - c_X + n_Y - c_Y - (|V(G)| - 1) + 1 \\ &\leq n_X - 1 + n_Y - 1 - (|V(G)| - 1) + 1 \\ &= |V(G_X) \cap V(G_Y)| \,. \end{aligned}$$

Clearly, the inequality becomes an equality if and only if $c_X = c_Y = 1$, i.e., both G_X and G_Y are connected. □

We now introduce, inspired by Proposition 5.16, a notion of a k-separation. Let $\mathcal{M}$ be a matroid with a ground set E. A partition (X, Y) of E is called a *separation*. A separation (X, Y) is a k-separation if

$$r(X) + r(Y) - r(E) \leq k - 1 \,.$$

A separation (X, Y) is a *Tutte k-separation* if it is a k-separation and

$$\min\{|X|, |Y|\} \geq k \,.$$

A separation (X, Y) is a *cyclic k-separation* if it is a k-separation and

$$r(X) < |X| \text{ and } r(Y) < |Y| \,.$$

A separation (X, Y) is a *essential k-separation* if it is a k-separation and

$$\min\{r(X), r(Y)\} \geq k \,.$$

Observe that every essential k-separation is also a Tutte k-separation.

The *(Tutte) connectivity* of a matroid $\mathcal{M}$ is the minimum k such that $\mathcal{M}$ has a Tutte k-separation and is denoted by $\lambda(\mathcal{M})$. If $\mathcal{M}$ has no Tutte k-separation, then $\lambda(\mathcal{M})$ is set to be equal to ∞. Observe that a matroid $\mathcal{M}$ is connected if and only if $\lambda(\mathcal{M}) \geq 2$. In general, we define a matroid $\mathcal{M}$ to be *k-connected*, $k \geq 2$, if $\lambda(\mathcal{M}) \geq k$.

Similarly, the *cyclic connectivity* of a matroid $\mathcal{M}$ is the minimum k such that $\mathcal{M}$ has a cyclic k-separation and is denoted by $\gamma(\mathcal{M})$ and $\mathcal{M}$ is *cyclically k-connected*, if $\gamma(\mathcal{M}) \geq k$. Finally, the *essential connectivity* of a matroid $\mathcal{M}$ is the minimum k such that $\mathcal{M}$ has an essential k-separation and is denoted by $\kappa(\mathcal{M})$ and $\mathcal{M}$ is *essentially k-connected*, if $\kappa(\mathcal{M}) \geq k$. For summary, see Table 5.1.

We will first address several basic properties of the just defined notions and we then relate them to the analogous notions for graphs. Let us start with the relation between the connectivity of a matroid and its dual.

Type of k-separation (X, Y)	Condition
Tutte	$\min\{\lvert X\rvert, \lvert Y\rvert\} \geq k$
Cyclic	$r(X) < \lvert X\rvert$ and $r(Y) < \lvert Y\rvert$
Essential	$\min\{r(X), r(Y)\} \geq k$

Table 5.1: An overview of different kind of separations in matroids.

Proposition 5.17. *The connectivity of a matroid $\mathcal{M}$ and its dual $\mathcal{M}^*$ is the same.*

Proof. By Proposition 2.4, it holds for every subset $X \subseteq E(\mathcal{M})$ that

$$\begin{aligned} r(X) + r(E \setminus X) - r(E) &= r(X) + r^*(X) - |X| \\ &= r^*(X) + r^*(E \setminus X) - r^*(E)\,. \end{aligned}$$

In particular, $(X, E \setminus X)$ is a Tutte k-separation of $\mathcal{M}$ if and only if it is a Tutte k-separation of $\mathcal{M}^*$. □

Let us now look at the other two connectivity parameters.

Proposition 5.18. *Let $\mathcal{M}$ be a matroid. The essential connectivity of the matroid $\mathcal{M}$ is equal to the cyclic connectivity of the dual matroid $\mathcal{M}^*$.*

Proof. Let (X, Y) be an essential k-separation of $\mathcal{M}$. As in the proof of Proposition 5.17, we can argue that (X, Y) is a k-separation of $\mathcal{M}^*$. Since (X, Y) is essential, $r(X) \geq k$ which implies, using the fact that (X, Y) is a k-separation, that $r(Y) \leq r(\mathcal{M}) - 1$. Hence, X contains a circuit of $\mathcal{M}^*$. Symmetrically, Y contains a circuit of $\mathcal{M}^*$. We conclude that (X, Y) is a cyclic k-separation of $\mathcal{M}^*$.

In the other direction, if (X, Y) is a cyclic k-separation of $\mathcal{M}^*$ for k as small as possible, then $r(Y) \leq r(\mathcal{M}) - 1$ and $r(X) \geq k$. Similarly, $r(Y) \geq k$ and (X, Y) an essential k-separation of $\mathcal{M}$. □

The cyclic connectivity of a matroid $\mathcal{M}$ is sometimes denoted by $\kappa^*(\mathcal{M})$ (instead of $\gamma(\mathcal{M})$) inspired by the equality stated in Proposition 5.18 which can then be written as $\kappa^*(\mathcal{M}) = \kappa(\mathcal{M}^*)$.

We next show that k-connected matroids are free of small circuits and cocircuits.

Proposition 5.19. *If $\mathcal{M}$ is an k-connected matroid with at least $2(k-1)$ elements, then all circuits and all cocircuits of $\mathcal{M}$ have at least k elements.*

Proof. Let C be a circuit of $\mathcal{M}$ with $k' < k$ elements. Observe that $|E \setminus C| \geq k'$ where E is the ground set of $\mathcal{M}$. Since $r(C) + r(E \setminus C) \leq r(E) + k' - 1$, the partition $(C, E \setminus C)$ is a k'-separation of $\mathcal{M}$ contradicting the fact that $\mathcal{M}$ is k-connected.

We now show that $\mathcal{M}$ has no cocircuits with less than k elements. By Proposition 5.17, $\mathcal{M}^*$ is k-connected and the already established part of the proposition implies that $\mathcal{M}^*$ has no circuits with less than k elements. Hence, $\mathcal{M}$ has no cocircuits with less than k elements. □

An argument from the proof of Proposition 5.19 also implies the following.

Proposition 5.20. *If a matroid $\mathcal{M}$ is k-connected and has at least $2k-1$ elements, then $\mathcal{M}$ has no k-element subset that is both a circuit and a cocircuit.*

It is easy to see that the connectivity of a k-connected matroid with less than $2k$ elements is infinite. This and Proposition 5.19 allows us to determine the connectivity of uniform matroids and show that they are the only matroids with infinite connectivity:

Proposition 5.21. *The connectivity of a uniform matroid $U_{r,n}$ is given by the following formula:*

$$\lambda(U_{r,n}) = \begin{cases} r+1 & \text{if } n \geq 2r+2\,, \\ n-r+1 & \text{if } n \leq 2r-2\,, \\ \infty & \text{otherwise.} \end{cases}$$

Moreover, if the connectivity of a matroid $\mathcal{M}$ is infinite, then $\mathcal{M}$ is uniform.

Proof. By Proposition 5.17, we can assume that $2r \leq n$. Let (X, Y) be a Tutte k-separation of $U_{r,n}$. By symmetry, we can assume that $|X| \leq |Y|$. In particular, $r(Y) = r$. Hence, if $r(X) + r(Y) - r(X \cup Y) = r(X) \leq k-1$ and $|X| \geq k$, then $k - 1 \geq r$. This implies that the connectivity of $U_{r,n}$ is at least $r+1$, i.e., $U_{r,n}$ is $(r+1)$-connected. Since $U_{r,n}$ contains a circuit with $r+1$ elements, the connectivity of $U_{r,n}$ is equal to $r+1$ by Proposition 5.19 if $n \geq 2r+2$. Otherwise, the assumption $2r \leq n$ implies that $n \in \{2r, 2r+1\}$. It is easy to verify that the connectivity of matroids $U_{r,2r}$ and $U_{r,2r+1}$ is infinite.

Assume now that $\mathcal{M}$ is a matroid with infinite connectivity and let n be the number of its elements. We can assume that the rank of $\mathcal{M}$ is at most $n/2$ by Proposition 5.17. Assume that n is even. By Proposition 5.19, every circuit of $\mathcal{M}$ has at least $n/2+1$ elements, i.e., all subsets of the ground set of $\mathcal{M}$ with less than $n/2+1$ elements are independent. Since the rank of $\mathcal{M}$ is at most $n/2$, $\mathcal{M}$ is isomorphic to $U_{n/2,n}$. If n is odd, we conclude that every circuit has at least $n/2 + 1/2$ elements and thus $\mathcal{M}$ must be isomorphic to $U_{n/2-1/2,n}$. □

We now relate the connectivity of a matroid to its essential and cyclic connectivity.

Proposition 5.22. *Let $\mathcal{M}$ be a matroid with finite connectivity and let k be an integer. The matroid $\mathcal{M}$ is k-connected if and only if it is both essentially and cyclically k-connected.*

Proof. Since the connectivity of $\mathcal{M}$ is finite, $\lambda(\mathcal{M}) \le |E(\mathcal{M})|/2$. Since every essential k-separation is also a Tutte k-separation, if $\mathcal{M}$ is k-connected, then $\mathcal{M}$ is also essentially k-connected. If (X, Y) is a cyclic k-separation, then (X, Y) is a Tutte k-separation unless $|X| < k$ or $|Y| < k$. In the latter case, if $|X| < k$, then $\mathcal{M}$ has a circuit C with at most $k-1$ elements and $(C, E(\mathcal{M}) \setminus C)$ is a Tutte k'-separation with $k' \le k-1$. In both cases, $\mathcal{M}$ contains a Tutte k'-separation for some $k' < k$. This shows that if $\mathcal{M}$ is k-connected, then $\mathcal{M}$ is also cyclically k-connected.

Assume now that $\mathcal{M}$ is both essentially and cyclically k-connected though it has a Tutte k'-separation (X, Y) with $k' < k$. Choose such a Tutte k'-separation with k' as small as possible. In particular, $r(X) + r(Y) - r(\mathcal{M}) = k' - 1$ which implies that both $r(X)$ and $r(Y)$ are at least $k' - 1$.

Since (X, Y) is not a cyclic k'-separation, at least one of the sets X and Y is independent, say X. In particular, $k' \le |X| = r(X)$. Since (X, Y) is not an essential k'-separation, $r(Y) < k'$ which implies $r(Y) = k' - 1$. By moving some elements from Y to X, we can assume that Y is a circuit with k' elements. If X was not an independent set after adding these elements, we would obtain a cyclic k'-separation.

Consider an arbitrary element x of X. Let $X' = X - x$ and $Y' = Y + x$. By the minimality of k', (X', Y') is not a Tutte $(k'-1)$-separation and

$$k' - 1 \le r(X') + r(Y') - r(\mathcal{M}) = |X| - 1 + r(Y') - r(\mathcal{M}) \le r(Y) = k' - 1\,.$$

Comparing the left-hand and right-hand sides yields that all the inequalities are equalities and thus $r(Y') = r(Y) + 1 = k'$. The rank of X' is equal to $|X| - 1 = |E(\mathcal{M})| - k' - 1$ which is at least k' since $k' < k \le |E(\mathcal{M})|/2$. This implies (X', Y') is an essential k-separation. So, we can conclude that $\mathcal{M}$ has no Tutte k'-separation with $k' < k$, i.e., $\mathcal{M}$ is k-connected. □

We now turn to relating connectivity of graphic matroids to connectivity of associated graphs. We start with essential connectivity of graphic matroids. Recall that a graph G is *k-connected* if it has at least $k+1$ vertices and it stays connected after removing any at most $k-1$ vertices.

Theorem 5.23. *Let G be a connected graph with at least $k+1$ vertices, $k \ge 2$. The graph G is k-connected if and only if the matroid $\mathcal{M}(G)$ is essentially k-connected.*

Proof. Assume that G is not k-connected, i.e., there exists a subset S, $|S| < k$, of its vertices such that G becomes disconnected after removing the vertices of S. Choose such an inclusion-wise minimal set S. Let A be the vertex set of one of the components of $G \setminus S$ and let $B = V(G) \setminus (A \cup S)$. Observe that the subgraph G_A with vertex set $A \cup S$ that contains all the edges with both end-vertices in $A \cup S$ and the subgraph G_B with vertex $B \cup S$ that contains all the edges incident with a vertex of B are connected (otherwise, there would be a proper subset S' of S such that $G \setminus S'$ is not connected). Since $A \neq \emptyset$ and $B \neq \emptyset$, the rank of each of $E(G_A)$ and $E(G_B)$ in $\mathcal{M}$ is at least $|S|$. By Proposition 5.16, $(E(G_A), E(G_B))$ is a $|S|$-separation and since $|E(G_A)| \geq |S|$ and $|E(G_B)| \geq |S|$, it is an essential $|S|$-separation. This shows that if $\mathcal{M}$ is essentially k-connected, then G is k-connected.

Suppose that G is k-connected but $\mathcal{M}$ has an essential k'-separation (E_1, E_2) for some $k' < k$. Let G_i be the spanning subgraph of G formed by edges of E_i, $i = 1, 2$. Assume that there exist vertices u and v lying in different components of G_1 as well as different components of G_2. Since G is k-connected, Menger's theorem implies that there exist internally vertex-disjoint paths $P_1, P_2, \ldots, P_k$ connecting u and v. Let A_i denote $E_i \cap (\cup_{j=1}^{n} E(P_j))$, $i = 1, 2$. Clearly, neither A_1 nor A_2 contains a circuit of $\mathcal{M}$ (otherwise, it would correspond to a cycle of G containing both u and v contradicting that u and v lie in different components of G_i, $i = 1, 2$). Hence, both A_1 and A_2 are independent and it holds that

$$r(A_1) + r(A_2) - r(A_1 \cup A_2) = |A_1| + |A_2| - (|A_1 \cup A_2| - k + 1) = k - 1 \geq k' .$$

When adding one edge of $E \setminus (A_1 \cup A_2)$ after another, we observe that the left-hand side of the above inequality either does not change or increases by one. In particular, it holds that

$$r(E_1) + r(E_2) - r(E) \geq r(A_1) + r(A_2) - r(A) \geq k'$$

which is impossible since (E_1, E_2) is a k'-separation.

We have just shown that any pair of vertices is contained in the same component in at least one of the graphs G_1 or G_2 which implies that at least one of the graphs G_1 or G_2 is connected, say G_1. Since G_1 is connected and spanning, we obtain that $r(E_1) = r(E)$ where E is the ground set of $\mathcal{M}$. Since (E_1, E_2) is a k'-separation, it holds that $r(E_1) + r(E_2) - r(E) \leq k' - 1$, but since it is an essential k'-separation, it holds that $r(E_1) + r(E_2) - r(E) = r(E_2) \geq k'$ which is impossible. This finishes the proof of the theorem. □

We next address cyclic connectivity of matroids. Let us recall the notion of cyclic connectivity for graphs: let G be a graph and E_1 and E_2 a partition of its edge set. We say that (E_1, E_2) is a *cyclic k-cut* if there are only k vertices incident with edges both from E_1 and E_2 and G contains a cycle formed by edges of each of E_1 and E_2. A graph G is *cyclically k-connected* if it has no cyclic k'-cut for $k' < k$.

Theorem 5.24. *Let G be a connected graph and let $k \geq 2$. The graph G is cyclically k-connected if and only if $\mathcal{M}(G)$ is cyclically k-connected.*

Proof. Any cyclic k'-cut of G, $k' < k$, corresponds to a cyclic k'-separation of $\mathcal{M}(G)$ by Proposition 5.16. In particular, if $\mathcal{M}(G)$ is cyclically k-connected, then G is also cyclically k-connected. Assume now that G is cyclically k-connected but $\mathcal{M}(G)$ has a cyclic k'-separation (E_1, E_2) for $k' < k$. Let G_i, $i = 1, 2$, be the subgraph of G formed by edges of E_i and vertices incident with them. Among all cyclic k'-separations (E_1, E_2) with $k' < k$ choose such that the total number of components of G_1 and G_2 is as small as possible.

If both G_1 and G_2 are connected, then (E_1, E_2) is a cyclic k'-cut of G by Proposition 5.16. Hence, at least one of the graphs G_1 and G_2 is not connected, say G_2. Let H be a component of G_2 that does not contain all cycles of G_2. Set $E_1' = E_1 \cup E(H)$ and $E_2' = E_2 \setminus E(H)$. Clearly, both E_1' and E_2' contain a circuit of $\mathcal{M}(G)$. Since H is a component of G_2, we obtain that

$$\begin{aligned} r(E_1') + r(E_2') &\le (r(E_1) + |V(H)| - 1) + (r(E_2) - |V(H)| + 1) \\ &= r(E_1) + r(E_2) \le r(E) + k - 1 \end{aligned}$$

which implies that (E_1', E_2') is a cyclic k'-separation of $\mathcal{M}$. Since H was a component of G_2, the subgraphs G_1' and G_2' corresponding to (E_1', E_2') have a smaller number of components than G_1 and G_2 which contradicts their choice. $\square$

It remains to consider an analogue of Tutte connectivity. Let us define the analogous notion for graphs: A partition (E_1, E_2) of the edge set of a graph G is a *Tutte k-cut* if there are only k vertices incident with edges both from E_1 and E_2 and both E_1 and E_2 contain at least k edges each. A graph G is *Tutte k-connected* if it has no Tutte k'-cut for $k' < k$.

Theorem 5.25. *Let G be a connected graph and let $k \ge 2$. The graph G is Tutte k-connected if and only if the matroid $\mathcal{M}(G)$ is k-connected.*

Proof. As in the proof Theorem 5.24, it can be shown that Proposition 5.16 implies that any Tutte k'-cut of G, $k' < k$, corresponds to a Tutte k'-separation of $\mathcal{M}(G)$. In particular, if $\mathcal{M}(G)$ is k-connected, then G is also Tutte k-connected. Suppose now that G is Tutte k-connected. This implies that G is k-connected which implies that $\mathcal{M}(G)$ is essentially k-connected by Theorem 5.23. If $\mathcal{M}(G)$ is not k-connected, then $\mathcal{M}(G)$ has a cyclic k'-separation, $k' < k$, by Proposition 5.22. Since such a cyclic k'-separation is not essential, $\mathcal{M}(G)$ must contain a circuit of size k' (and we can assume $k' \le |E(G)|/2$ by considering the part of the k'-separation with smaller rank). However, such a circuit corresponds to a cycle C of G of length k' and $(E(C), E(G) \setminus E(C))$ is a Tutte k'-cut of G since $|E(G)| \ge 2k'$. $\square$

Inspired by the notion of the girth of a graph, which is the length of its shortest cycle, we define the *girth* $g(\mathcal{M})$ of a matroid $\mathcal{M}$ to be the number of elements of its smallest circuit. If $\mathcal{M}$ has no circuits, we set $g(\mathcal{M}) = \infty$. We relate the connectivity of a matroid to its essential connectivity and its girth.

Theorem 5.26. *Let $\mathcal{M}$ be a matroid that is not isomorphic to any uniform matroid $U_{r,n}$ with $n \geq 2r - 1$. The connectivity of $\mathcal{M}$ is then given by*

$$\lambda(\mathcal{M}) = \min\{\kappa(\mathcal{M}), g(\mathcal{M})\} .$$

Proof. By Proposition 5.21, $\lambda(\mathcal{M})$ is finite. The definition of the connectivity of a matroid yields that $\mathcal{M}$ has at least $2\lambda(\mathcal{M})$ elements which implies that the girth of $\mathcal{M}$ is at least $\lambda(\mathcal{M})$ by Proposition 5.19. Proposition 5.22 yields that the essential connectivity of $\mathcal{M}$ is at least $\lambda(\mathcal{M})$. This establishes that $\lambda(\mathcal{M}) \leq \min\{\kappa(\mathcal{M}), g(\mathcal{M})\}$.

To prove the other direction, let (X, Y) be a Tutte k-separation for $k = \lambda(\mathcal{M})$. If (X, Y) is an essential k-separation, then the equality holds. Hence, we assume that (X, Y) is not essential, i.e., $r(X) \leq k - 1$ which implies that $\mathcal{M}$ has a circuit with at most k elements, i.e., $g(\mathcal{M}) \leq k$. Again, the inequality is an equality. □

We now translate Theorem 5.26 to graphs.

Corollary 5.27. *Let G be a connected graph with at least 3 vertices that is not isomorphic to K_3. The connectivity of $\mathcal{M}(G)$ is equal to the minimum of the connectivity and the girth of G.*

Proof. As G is connected and $|V(G)| \geq 3$, we have $r(\mathcal{M}(G)) \geq 2$. Since $U_{2,4}$ is not graphic and the class of graphic matroids is closed under taking minors, $U_{r,n}$ is not graphic if both r and $n - r$ exceed 1. If $\mathcal{M}(G)$ is isomorphic to $U_{r,n}$, then either $r = 1$ and G is a tree or $n = r + 1$ and G is a cycle of length n. Clearly, the statement holds unless G is isomorphic K_3 which is forbidden by the assumptions. The corollary now follows from Theorem 5.23 and Theorem 5.26 unless G is a complete graph (in which case Theorem 5.23 cannot be applied). In such case, both the connectivity of $\mathcal{M}(G)$ and the girth of G are equal to three unless G is a complete graph with four vertices. □

Corollary 5.27 readily yields the following.

Corollary 5.28. *Let G be a simple graph with at least three vertices that has no isolated vertices and is not isomorphic to a complete graph of order three. The matroid $\mathcal{M}(G)$ is 3-connected if and only if G is 3-connected and simple.*

5.3 Computing minimal separations

A natural question from the algorithmic point of view is how efficiently we can find a small separation splitting two given sets of elements of a matroid. To be more precise, for two sets F_1 and F_2 of elements of a matroid $\mathcal{M}$, a partition (E_1, E_2) of the ground set of $\mathcal{M}$ to two sets E_1 and E_2 is an (F_1, F_2)-*separation* if

$F_i \subseteq E_i$, $i \in \{1,2\}$. We show that a minimal (F_1,F_2)-separation can be computed in polynomial time. The presented algorithm will also play a crucial role when constructing optimal branch-decompositions in Chapter 8.

Theorem 5.29. *There is an algorithm running in time $O(r^2 n\tau)$ that determines for an oracle-given matroid $\mathcal{M}$ and two disjoint sets F_1 and F_2 of its elements whether there is an (F_1,F_2)-separation that is k-separation, where r is the rank $\mathcal{M}$, n is the number of elements of $\mathcal{M}$ and τ is the oracle query time. Moreover, such an (F_1,F_2)-separation for the smallest possible value k can be found within the same time estimate.*

Proof. The goal is to find a minimal integer k such that there exists a partition (E_1,E_2) of the ground set E of $\mathcal{M}$ satisfying

$$r_{\mathcal{M}}(E_1) + r_{\mathcal{M}}(E_2) = r_{\mathcal{M}}(E) + k$$

with $F_i \subseteq E_i$, $i \in \{1,2\}$. This formula resembles the right hand side of the equality in Theorem 3.4 but we have to find suitable matroids to apply the Matroid Intersection Theorem in our setting.

Let $\mathcal{M}_i$ be the matroid obtained from $\mathcal{M}$ by contracting the elements of F_i and then removing the elements of the other set F_{3-i}. The matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ have the same ground set E_0 which is $E \setminus (F_1 \cup F_2)$. Using the definition of matroid contraction, we obtain the following:

$$\min_{E_1 \cup E_2 = E_0} r_{\mathcal{M}_1}(E_1) + r_{\mathcal{M}_2}(E_2) =$$

$$\left(\min_{E_1 \cup E_2 = E_0} r_{\mathcal{M}}(E_1 \cup F_1) + r_{\mathcal{M}}(E_2 \cup F_2)\right) - r_{\mathcal{M}}(F_1) - r_{\mathcal{M}}(F_2)\ . \tag{5.1}$$

By the Matroid Intersection Theorem and (5.1), the minimum k such that there is a (F_1,F_2)-separation that is also a k-separation is equal to $K + r_{\mathcal{M}}(F_1) + r_{\mathcal{M}}(F_2) - r_{\mathcal{M}}(E) + 1$ where K is the maximum size of a subset of E_0 independent both in $\mathcal{M}_1$ and $\mathcal{M}_2$. By Corollary 3.5, this k can be determined in time claimed in the statement of the theorem. Moreover, a partition (E_1,E_2) of E_0 for which the minimum in (5.1) is attained can also be found in time $O(r^2 n\tau)$. A partition $(E_1 \cup F_1, E_2 \cup F_2)$ is an (F_1,F_2)-separation of $\mathcal{M}$ that is a k-separation of $\mathcal{M}$ for the smallest possible value of k. $\square$