Chapter 2

Duality and minors

In this chapter, we introduce two concepts for matroids that have their counterparts in graph theory. The first one is the concept of dual matroids, generalizing the concept of dual graphs for graphs drawn in the plane, and the second one is the concept of minors.

2.1 Dual matroids

Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid and let $\mathcal{B}$ be the family of bases of $\mathcal{M}$. Let $\mathcal{B}^*$ the family of the complements of the members of $\mathcal{B}$, i.e., $X \in \mathcal{B}^*$ if and only if $E \setminus X \in \mathcal{B}$. We show that $\mathcal{B}^*$ is a family of bases of a matroid.

Theorem 2.1. *Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid and let $\mathcal{B}$ be the family of bases of $\mathcal{M}$. The family $\mathcal{B}^*$ is a family of bases of a matroid on E.*

Proof. We first prove that the family $\mathcal{B}$ has the following property:

(B2)' for every $B_1, B_2 \in \mathcal{B}$ and $e \in B_2 \setminus B_1$, there is an element $f \in B_1 \setminus B_2$ such that $(B_1 - f) + e \in \mathcal{B}$.

Note the difference between (B2) and (B2)' in the quantification of e and f.

Since B_1 is a base, the property (C3) implies that $B_1 + e$ contains a unique circuit C containing e. Since C is dependent and B_2 is independent, $C \setminus B_2$ is non-empty and thus there exists $f \in C \setminus B_2$. Clearly, $f \in B_1 \setminus B_2$. Moreover, since $(B_1 - f) + x$ does not contain the circuit C and C is the unique circuit in $B_1 + e$, the set $(B_1 - f) + e$ must be independent. Finally, the fact $|(B_1 - f) + e| = |B_1|$ implies that $(B_1 - f) + e$ is a base.

We can now prove the theorem. As $\mathcal{B}(\mathcal{M})$ is non-empty, $\mathcal{B}^*(\mathcal{M})$ is non-empty, i.e., (B1) holds for $\mathcal{B}^*(\mathcal{M})$. Consider two members B_1^* and B_2^* of the family $\mathcal{B}^*$ and an element $x \in B_1^* \setminus B_2^*$. For $i = 1, 2$, let $B_i = E \setminus B_i^*$. Observe that $B_1^* \setminus B_2^* = B_2 \setminus B_1$ and thus $e \in B_2 \setminus B_1$. By (B2)', there exists an element $f \in B_1 \setminus B_2 = B_2^* \setminus B_1^*$ such that $(B_1 - f) + e$ is a base of $\mathcal{M}$. Consequently, and $E \setminus ((B_1 - f) + e) = (B_1^* - e) + f$ is in the family $\mathcal{B}^*$. We conclude that the family $\mathcal{B}^*$ satisfies both (B1) and (B2). $\square$

The matroid described in Theorem 2.1 is *the dual* of $\mathcal{M}$ and is denoted by $\mathcal{M}^*$. Observe that $\mathcal{B}(\mathcal{M}^*) = \mathcal{B}^*(\mathcal{M})$. The definition of the notion of dual matroids directly yields that

Proposition 2.2. *The dual of the dual of a matroid $\mathcal{M}$ is the matroid $\mathcal{M}$ itself, i.e., $(\mathcal{M}^*)^* = \mathcal{M}$.*

As an example, let us consider a uniform matroid $U_{m,n}$. Its bases are all of the m-element subsets of $E(U_{m,n})$ and hence the bases of $U^*_{m,n}$ are all $(n-m)$-element subsets of the ground set, i.e. $U^*_{m,n} = U_{n-m,n}$. We have now seen that the class of uniform matroids is closed under taking duals.

The rank function of the dual matroid is usually denoted by r^* and $r^*(\mathcal{M}^*)$ denotes the rank of the dual matroid. The function r^* is also referred as to the *corank function* of $\mathcal{M}$. Using the definition of bases of the dual matroid and the fact that matroid and its dual are both on the same ground set, one can observe.

Proposition 2.3. $r(\mathcal{M}) + r^*(\mathcal{M}^*) = |E(\mathcal{M})| = |E(\mathcal{M}^*)|$.

In fact, we can generalize this observation to obtain an explicit formula for the corank function.

Proposition 2.4. *For every subset X of the ground set E of a matroid $\mathcal{M}$, it holds that*

$$r^*(X) = |X| - r(\mathcal{M}) + r(E - X)\,.$$

Proof. Let I^* be an inclusion-wise maximal independent subset of X in $\mathcal{M}^*$ and let I be an inclusion-wise maximal subset of $E \setminus X$ independent in $\mathcal{M}$, i.e., $r^*(X) = |I^*|$ and $r(E \setminus X) = |I|$. Further, let B be an inclusion-wise maximal independent subset of $E \setminus I^*$ that contains I. Since $r(B) = r(E \setminus I^*)$ and $r(E \setminus I^*) = r(\mathcal{M})$, B is a base of $\mathcal{M}$.

Let $B^* = E \setminus B$; since B is a base of $\mathcal{M}$, B^* is a base of $\mathcal{M}^*$. Clearly, $I^* \subseteq B^*$ and $B^* \cap X = I^*$. Similarly, $I \subseteq B$ and $B \cap (E \setminus X) = I$. In particular, $|B \cap X| = |B| - |I|$ and thus

$$|X| = |X \cap B| + |X \cap B^*| = |B| - |I| + |I^*| = r(\mathcal{M}) - r(E - X) + r^*(X)\,.$$

$\square$

Loops of $\mathcal{M}^*$ are called *coloops*. Observe that an element of $\mathcal{M}$ is a coloop if and only if it is contained in every base of $\mathcal{M}$. The family of circuits of the dual matroids $\mathcal{C}(\mathcal{M}^*)$ is denoted by $\mathcal{C}^*(\mathcal{M})$ and the members of this family are called *cocircuits* of $\mathcal{M}$. We characterize subsets of the ground sets that are cocircuits in the next lemma.

Lemma 2.5. *Let $\mathcal{M}$ be a matroid on a set E. A subset C^* of E is a cocircuit of $\mathcal{M}$ if and only if $E \setminus C^*$ is a hyperplane of $\mathcal{M}$.*

Proof. Let C^* be a cocircuit of $\mathcal{M}$. We apply Proposition 2.4 and the definition of the dual matroid to obtain that

$$\begin{aligned} r(E \setminus C^*) &= |E \setminus C^*| - r^*(\mathcal{M}) + r^*(C^*) \\ &= |E| - |C^*| - r^*(\mathcal{M}) + |C^*| - 1 \\ &= r(\mathcal{M}) - 1\,. \end{aligned}$$

Since C^* is an inclusion-wise minimal set with $r^*(C^*) = |C^*| - 1$, the set $E - C^*$ is an inclusion-wise maximal subset of E with rank $r(\mathcal{M}) - 1$ and thus it is a hyperplane of $\mathcal{M}$. The opposite direction can be proved along the same lines. $\square$

We next see that a circuit and a cocircuit cannot intersect at exactly one element.

Proposition 2.6. *If C is a circuit and C^* is a cocircuit of the matroid $\mathcal{M}$, then $|C \cap C^*| \neq 1$.*

Proof. Assume that there exists a circuit C and a cocircuit C^* such that $C \cap C^* = \{e\}$. By Lemma 2.5, $H = E \setminus C^*$ is a hyperplane of $\mathcal{M}$ and $e \notin H$ by the choice of C and C^*. This however contradicts the submodularity of the rank function:

$$r(C) + r(\mathcal{M}) = r(C - e) + r(H + e) = r(C \cap H) + r(C \cup H) \leq r(C) + r(H)$$

which is impossible since the rank of H is $r(\mathcal{M}) - 1$. $\square$

We finish this section with an observation on the dual of a union of two matroids.

Proposition 2.7. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two matroids with disjoint ground sets. The dual of $\mathcal{M}_1 \oplus \mathcal{M}_2$ is equal to $\mathcal{M}_1^* \oplus \mathcal{M}_2^*$.*

Proof. Let E_i be the ground set of $\mathcal{M}_i$, $i = 1, 2$, and let $E = E_1 \cup E_2$. In order to prove the statement of the proposition, we show that the families of bases of $(\mathcal{M}_1 \oplus \mathcal{M}_2)^*$ and $\mathcal{M}_1^* \oplus \mathcal{M}_2^*$ coincide. To this end, we apply Proposition 1.21.

$$\begin{aligned} \mathcal{B}((\mathcal{M}_1 \oplus \mathcal{M}_2)^*) &= \{B^* \mid E \setminus B^* \in \mathcal{B}(\mathcal{M}_1 \oplus \mathcal{M}_2)\} \\ &= \{E \setminus B \mid B \in \mathcal{B}(\mathcal{M}_1 \oplus \mathcal{M}_2)\} \\ &= \{E \setminus (B_1 \cup B_2) \mid B_1 \in \mathcal{B}(\mathcal{M}_1), B_2 \in \mathcal{B}(\mathcal{M}_2)\} \\ &= \{(E_1 \setminus B_1) \cup (E_2 \setminus B_2) \mid B_1 \in \mathcal{B}(\mathcal{M}_1), B_2 \in \mathcal{B}(\mathcal{M}_2)\} \\ &= \{B_1^* \cup B_2^* \mid B_1^* \in \mathcal{B}(\mathcal{M}_1^*), B_2^* \in \mathcal{B}(\mathcal{M}_2^*)\} \\ &= \mathcal{B}(\mathcal{M}_1^* \oplus \mathcal{M}_2^*)\,. \end{aligned}$$

$\square$

2.2 Matroid minors

Minors of matroids are defined using two basic operations, a deletion and a contraction of elements and their sets. Let $\mathcal{M}$ be a matroid on a ground set E and let T be a subset of the ground set E. The matroid obtained by *deleting* the subset T is the matroid with the ground set $E \setminus T$ whose independent sets are those subsets of $E \setminus T$ that are independent in $\mathcal{M}$. The matroid obtained by deleting T is denoted by $\mathcal{M} \setminus T$ or by $\mathcal{M}|(E \setminus T)$.

The matroid obtained by *contracting* of a set T is defined through deleting T in the dual matroid: the matroid $\mathcal{M}/T$ obtained by contracting T is the matroid $(\mathcal{M}^* \setminus T)^*$. Clearly, the ground set of $\mathcal{M}/T$ is $E \setminus T$. We like to remark that some authors do not allow deleting sets containing coloops and contracting sets containing loops. However, we do not observe this restriction here.

Let us now give formulas for the rank function of a matroid obtained by deleting or contracting a set of elements.

Proposition 2.8. *Let $\mathcal{M}$ be a matroid on a ground set E. For every subset T of E and every subset X of $E \setminus T$, the following holds:*

$$\begin{aligned} r_{\mathcal{M}\setminus T}(X) &= r_{\mathcal{M}}(X) \text{ and} \\ r_{\mathcal{M}/T}(X) &= r_{\mathcal{M}}(X \cup T) - r_{\mathcal{M}}(T) \, . \end{aligned}$$

Proof. The first equality directly follows from the definition of the deletion. To prove the second equality, we use the identity $r^*(X) = |X| - r_{\mathcal{M}}(\mathcal{M}) + r_{\mathcal{M}}(E \setminus X)$ given by Proposition 2.4 for the corank function of a matroid (r^* will always denote the corank of $\mathcal{M}$ throughout this proof). In particular, the following equalities hold:

$$\begin{aligned} r_{\mathcal{M}/T}(X) &= |X| + r_{\mathcal{M}^*\setminus T}(E \setminus T \setminus X) - r_{\mathcal{M}^*\setminus T}(E \setminus T) \\ &= |X| + r^*(E \setminus (T \cup X)) - r^*(E \setminus T) \\ &= |X| + (|E \setminus (T \cup X)| + r_{\mathcal{M}}(T \cup X) - r_{\mathcal{M}}(E)) \\ &\quad - (|E \setminus T| + r_{\mathcal{M}}(T) - r_{\mathcal{M}}(E)) = \\ &= r_{\mathcal{M}}(T \cup X) - r_{\mathcal{M}}(T) \end{aligned}$$

Note that the last equality holds since $|X| + |E \setminus (X \cup T)| = |E \setminus T|$ as $X \subseteq E \setminus T$. □

We can now characterize independent sets, bases and circuits of a matroid obtained by contracting of a subset of its ground set.

Proposition 2.9. *Let $\mathcal{M}$ be a matroid with ground set E and T a subset of E. For every base B_T of a base of $\mathcal{M}|T$, it holds that*

$$\mathcal{I}(\mathcal{M}/T) = \{I \subseteq E \setminus T | I \cup B_T \in \mathcal{I}(\mathcal{M})\} \, .$$

Proof. Let I be a subset of $E-T$ such that $(I \cup B_T) \in \mathcal{I}(\mathcal{M})$. Since B_T is a base of $\mathcal{M}|T$, it holds that $r_{\mathcal{M}}(I \cup B_T) = r_{\mathcal{M}}(I \cup T)$. Proposition 2.8 now yields that

$$\begin{aligned} r_{\mathcal{M}/T}(I) &= r_{\mathcal{M}}(I \cup T) - r_{\mathcal{M}}(T) \\ &= r_{\mathcal{M}}(I \cup B_T) - r_{\mathcal{M}}(B_T) = |I \cup B_T| - |B_T| = |I| \,. \end{aligned}$$

On the other hand, if $X \in \mathcal{I}(\mathcal{M}/T)$, then

$$\begin{aligned} |X| = r_{\mathcal{M}/T}(X) &= r_{\mathcal{M}}(X \cup T) - r_{\mathcal{M}}(T) \\ &= r_{\mathcal{M}}(X \cup B_T) - |B_T| \,. \end{aligned}$$

Hence, $|X \cup B_T| = |X| + |B_T| = r_{\mathcal{M}}(X \cup B_T)$, i.e., X is of the form described in the statement of the proposition. □

Proposition 2.9 also allows us to characterize bases of a matroid obtained by contracting of a subset of its ground set.

Corollary 2.10. *Let $\mathcal{M}$ be a matroid with ground set E and T a subset of E. For every base B_T of a base of $\mathcal{M}|T$, the bases of $\mathcal{M}/T$ are precisely sets B' such that $B' \cup B_T$ are bases of $\mathcal{M}$.*

Finally, we characterize circuits of a matroid obtained by contraction.

Proposition 2.11. *Let $\mathcal{M}$ be a matroid with ground set E and T a subset of E. A subset of $E \setminus T$ is a circuit of $\mathcal{M}/T$ if and only if it is an inclusion-wise minimal non-empty member of the family of sets of the form $C \setminus T$ for $C \in \mathcal{C}(\mathcal{M})$.*

Proof. Consider a circuit $C_1 \in \mathcal{C}(\mathcal{M}/T)$. For an arbitrary base B_T of $\mathcal{M}|T$, it holds that $C_1 \cup B_T \notin \mathcal{I}(\mathcal{M})$ but $(C_1 - e) \cup B_T \in \mathcal{I}(\mathcal{M})$ for any $e \in C_1$. Hence, there exists a circuit D of $\mathcal{M}$ such that $C_1 \subseteq D \subseteq C_1 \cup B_T$ and we get $C_1 = D - T$.

Now suppose that $C_2 \setminus T$ is an inclusion-wise minimal non-empty member of the family of sets $C \setminus T$, $C \in \mathcal{C}(\mathcal{M})$. Clearly, $C_2 \cap T \subsetneq C_2$ and thus $C_2 \cap T \in \mathcal{I}(\mathcal{M})$. In particular, there exists a base B_T of $\mathcal{M}|T$ such that $C_2 \cap T \subseteq B_T$. As $C_2 \cup B_T$ contains C_2, the set $C_2 \cup B_T \notin \mathcal{I}(\mathcal{M})$ and thus $C_2 \setminus T \notin \mathcal{I}(\mathcal{M}/T)$. If $C_2 \setminus T \notin \mathcal{C}(\mathcal{M}/T)$, there would exist $C_3 \in \mathcal{C}(\mathcal{M}/T)$ and $C_3 \subsetneq C_2 \setminus T$. By the already established reverse implication, C_3 is equal to $D \setminus T$ for some circuit D of $\mathcal{M}$ which contradicts the choice of C_2. □

We next focus on the mutual relation of the deletion and contraction operation, in particular, we show that these operations commute.

Lemma 2.12. *Let T_1 and T_2 be two disjoint subsets of the ground set E of a matroid $\mathcal{M}$. The following holds:*

(i) $(\mathcal{M} \setminus T_1) \setminus T_2 = \mathcal{M} \setminus (T_1 \cup T_2) = (\mathcal{M} \setminus T_2) \setminus T_1$

(ii) $(\mathcal{M}/T_1)/T_2 = \mathcal{M}/(T_1 \cup T_2) = (\mathcal{M}/T_2)/T_1$

(iii) $(\mathcal{M}/T_1) \setminus T_2 = (\mathcal{M} \setminus T_2)/T_1$.

Proof. The part *(i)* directly follows from the definition of the deletion and the part *(ii)* is a consequence of the part *(i)* through considering the dual matroids. To prove *(iii)*, we show that $(\mathcal{M}/T_1) \setminus T_2$ and $(\mathcal{M} \setminus T_2)/T_1$ have the same rank function. If $X \subseteq E - (T_1 \cup T_2)$, then

$$\begin{aligned} r_{(\mathcal{M}/T_1)\setminus T_2}(X) &= r_{\mathcal{M}/T_1}(X) \\ &= r_{\mathcal{M}}(X \cup T_1) - r_{\mathcal{M}}(T_1) \\ &= r_{\mathcal{M}\setminus T_2}(X \cup T_1) - r_{\mathcal{M}\setminus T_2}(T_1) \\ &= r_{(\mathcal{M}\setminus T_2)/T_1}(X)\,. \end{aligned}$$

□

Lemma 2.12 implies that any sequence of deletions and contraction can be expressed as one contraction and one deletion, i.e., in the form $\mathcal{M} \setminus X/Y$ for some pair of disjoint subsets of the ground set. A matroid obtained from a matroid $\mathcal{M}$ that has this form is called a *minor* of $\mathcal{M}$. If $X \cup Y$ is non-empty, then the minor $\mathcal{M} \setminus X/Y$ is called *proper*. Matroid minors are closely related to graph minors as we shall see in Section 2.4. Let us finish this section with the following simple observation.

Proposition 2.13. *A matroid $\mathcal{N}$ is a minor of a matroid $\mathcal{M}$ if and only if $\mathcal{N}^*$ is a minor of $\mathcal{M}^*$.*

Proof. The statement immediately follows from the fact that $\mathcal{N} = \mathcal{M} \setminus X/Y$ if and only if $\mathcal{N}^* = \mathcal{M}^*/X \setminus Y$ for some disjoint subsets X and Y of the ground set of $\mathcal{M}$. □

2.3 Duality and minors for representable matroids

We now want to investigate how to obtain a representation of the dual of a matroid or a minor of it from its representation. Let us start with dual matroids.

Theorem 2.14. *Let $\mathcal{M}$ be a matroid and $[I_r|D]$ one of its standard representations. The matrix $\left[D^T|I_{n-r}\right]$ is one of the representations of $\mathcal{M}^*$ where the correspondence of the columns to the elements of the ground set in the two representations (given by their order) is the same (see Figure 2.1).*

$$\begin{array}{cc} e_1 \ldots e_r & e_{r+1} \ldots e_n \\ \multicolumn{2}{c}{\left(\begin{array}{c|c} \quad I_r \quad & \quad D \quad \end{array}\right)} \end{array} \qquad \begin{array}{cc} e_1 \;\ldots\; e_r & e_{r+1} \ldots e_n \\ \multicolumn{2}{c}{\left(\begin{array}{c|c} \quad D^T \quad & \quad I_{n-r} \quad \end{array}\right)} \end{array}$$

Figure 2.1: Representations of a matroid and its dual.

Proof. We show that the sets of $n-r$ linearly independent column vectors in $\left[D^T|I_{n-r}\right]$ are precisely bases of $\mathcal{M}^*$. Let B be an arbitrary set of r linearly independent columns of $\mathcal{M}[I_r|D]$, i.e., a base of $\mathcal{M}$. By permuting the columns if necessary, we may assume that $B = \{e_1, \ldots, e_s, e_{r+1}, \ldots, e_{r+(r-s)}\}$ for some $s \leq r$. Split now the two matrices as follows:

$$\begin{bmatrix} I_s & 0 & D_{11} & D_{12} \\ 0 & I_{r-s} & D_{21} & D_{22} \end{bmatrix} \longrightarrow \begin{bmatrix} D_{11}^T & D_{21}^T & I_{r-s} & 0 \\ D_{12}^T & D_{22}^T & 0 & I_{n-2r+s} \end{bmatrix}$$

Since B is a base, the rank of D_{21} is $r-s$ and thus the rank of D_{21}^T is also $r-s$. Hence, the submatrix of the latter matrix given by columns not in B has rank $(r-s)+(n-2r+s) = n-r$. In particular, this submatrix has full rank and thus its column vectors form a base of $\mathcal{M}^*$.

The presented argument can be reversed along the completely same lines to show that if $n-r$ column vectors are linearly independent in $\left[D^T|I_{n-r}\right]$, then the complementary r column vectors are linearly independent in $[I_r|D]$. Hence, the complement of a base B of $\mathcal{M}$ is independent in the matroid represented by $\left[D^T|I_{n-r}\right]$ and the complement of every inclusion-wise maximal independent set in the matroid represented by $\left[D^T|I_{n-r}\right]$ is a base of $\mathcal{M}$. We conclude that $\left[D^T|I_{n-r}\right]$ is a representation of $\mathcal{M}^*$.

$\square$

An immediate corollary of Theorem 2.14 is the following:

Corollary 2.15. *If a matroid $\mathcal{M}$ is representable over a field $\mathbb{F}$, then the dual matroid $\mathcal{M}^*$ is also representable over $\mathbb{F}$.*

Example: Consider the vector matroid of the following representation

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 & 0 \end{array}\right)$$

over GF(3). The dual matroid $\mathcal{M}^*$ is the vector matroid represented by the matrix

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 & 0 \end{array}\right).$$

Observe that this particular matroid, which is usually denoted by P_8, has the property that its dual is isomorphic to the matroid itself. Such matroids are called *self-dual*.

Let $\mathcal{M}$ be a matroid represented by a matrix A. If T is a set of columns of A, which corresponds to a subset of the ground set of $\mathcal{M}$, then $A \setminus T$ denotes the matrix obtained from A by removing the columns corresponding to T. Clearly, the definition of the deletion operation and representations of matroids yield the following:

Lemma 2.16. *Let $\mathcal{M}$ be a matroid represented by a matrix A and T a set of columns of A which we identify with a subset of the ground set of $\mathcal{M}$. The matrix $A \setminus T$ is a representation of the matroid $\mathcal{M} \setminus T$.*

Describing, the representation of a matroid obtained by a contraction of an element is more tricky.

Lemma 2.17. *Let $\mathcal{M}$ be a matroid and $[I_r|D]$ its standard representation. For every $1 \leq i \leq r$, the matrix obtained from $[I_r|D]$ by removing its i-th row is a representation of the matroid obtained from $\mathcal{M}$ by contracting its element corresponding to the i-th column of $[I_r|D]$.*

Proof. The lemma immediately follows from Lemma 2.16 and Theorem 2.14. □

Theorem 2.14 together with Lemmas 2.16 and 2.17 yields the following:

Corollary 2.18. *Every minor of an $\mathbb{F}$-representable matroid is $\mathbb{F}$-representable.*

In particular, the class of $\mathbb{F}$-representable matroids for a fixed field $\mathbb{F}$ is closed under taking minors by Corollary 2.18.

2.4 Duality and minors for graphic matroids

In this section, we investigate how minors of matroids are related to minors of graphs as well as duals of graphic matroids. Recall that $\mathcal{M}(G)$ denotes the graphic matroid corresponding to G. We start with matroids obtained by deleting or contracting some of the elements. Recall that $G \setminus T$ is a graph obtained by removing edges contained in T and G/T is the graph obtained by contracting edges contained in T (we do not remove arising loops and parallel edges).

Proposition 2.19. *Let G be a graph. The following holds for all subsets T of the edges of G:*

$$\mathcal{M}(G) \setminus T = \mathcal{M}(G \setminus T)$$

and

$$\mathcal{M}(G)/T = \mathcal{M}(G/T) \ .$$

Proof. It is enough to prove the statement for a single-element set $T = \{e\}$. The definitions of deletions of edges in graphs and elements in matroids directly yield that $\mathcal{M}(G) \setminus e = \mathcal{M}(G \setminus e)$. Contracting elements is more difficult. If e is a loop of G, then $G/e = G \setminus e$ and $\mathcal{M}(G)/e = \mathcal{M}(G) \setminus e$. Hence, we may assume that e is not a loop in G. A subset I of $E(G) - e$ is acyclic in G/e if and only if $I \cup e$ is acyclic in G. Hence, $\mathcal{I}(\mathcal{M}(G)/e) = \mathcal{I}(\mathcal{M}(G/e))$ and the proposition follows. $\square$

An immediate corollary of Proposition 2.19 is the following which implies that the class of graphic matroids is closed under taking minors.

Corollary 2.20. *Every minor of a graphic matroid is graphic.*

Let us turn our attention to duality of matroids and graphs. For a plane graph G, G^* denotes its geometric dual, i.e., the graph whose vertices are the faces of G and two of them are joined by an edge if they share an edge in G. In this way, the edges of G and G^* naturally one-to-one correspond. In particular, loops of G are bridges of G^* and bridges of G are loops of G^*.

Dual matroids of graphic matroid for plane graphs can be easily described as follows.

Lemma 2.21. *Let G be a plane connected graph. The matroid $\mathcal{M}(G^*)$ of the dual of G is isomorphic to the dual matroid $\mathcal{M}^*(G)$ of the graphic matroid $\mathcal{M}(G)$ corresponding to G.*

Proof. Through the natural one-to-one correspondence of the edge of G and G^*, we identify the elements of $\mathcal{M}(G^*)$ and $\mathcal{M}^*(G)$. Since $(G^*)^* = G$, it is enough to prove that for every spanning tree T of G, the edges not contained in T form a spanning tree of G^*.

Let $B \subseteq E$ be a subset of the edges of G forming a spanning tree in G. We first prove that $E \setminus B$ forms an acyclic subgraph of G^*. Assume that $E \setminus B$ contains a circuit, i.e., there is at least one vertex u of G inside this circuit and at least one vertex v of G outside. Clearly, any path connecting u and v in G have to intersect the considered circuit of G^* and thus it has to contain an edge of $E \setminus B$. Since B is a spanning tree of G, B must contain at least one such edge which is impossible. Hence, the edges of $E \setminus B$ form an acyclic subgraph of G^*.

We now apply Euler's formula that the edges of $E \setminus B$ forms a spanning tree of G^*. First, the number of vertices of G^* is the number of faces of G which is $2 + |E| - |V(G)|$. Since $|B| = |V(G)| - 1$ as the edges of B correspond to a spanning tree of G, $|E \setminus B| = |E| - |B| = |E| - |V(G)| + 1 = |V(G^*)| - 1$. Since the edges $E \setminus B$ form an acyclic subgraph of G^*, this subgraph must be connected. $\square$

Lemma 2.21 yields the following:

Theorem 2.22. *If G is a planar graph, then the dual of the matroid $\mathcal{M}(G)$ is graphic.*

We now prove that the circuits of the dual matroid of a graphic matroid corresponds to edge-cuts in a graph. Recall that *edge-cut* of a graph G is a set F of all edges such that the vertices of G can be partitioned into two sets A and A' such that each edge of F has exactly one end-vertex in A and one end-vertex in A'. Note that we use $\mathcal{M}^*(G)$ to denote the dual of the graphic matroid $\mathcal{M}(G)$ as we have already used in Lemma 2.21.

Lemma 2.23. *Let G be a graph. A subset F of the ground set of $\mathcal{M}^*(G)$ is a circuit in the dual matroid $\mathcal{M}^*(G)$ if and only if the edges of G corresponding to the elements of F form an inclusion-wise minimal edge-cut of G.*

Proof. Without loss of generality, we can assume that G is connected (identifying vertices of different components of G does not change $\mathcal{M}(G)$). We first show that the elements corresponding to an inclusion-wise minimal edge-cut of G form a circuit of $\mathcal{M}^*(G)$. Let A and B be a partition of the vertices of G and let F be the set of edges between A and B which we identify with elements of $\mathcal{M}(G)$ and thus of $\mathcal{M}^*(G)$. Assume that the edge-cut F is an inclusion-wise minimal, i.e., both the subgraph $G[A]$ of G induced by A and the subgraph $G[B]$ of G induced by B are connected. Every base of $\mathcal{M}(G)$ corresponds to a spanning tree of G and thus must include at least one edge from F, i.e., F is not independent in $\mathcal{M}^*(G)$. On the other hand, adding any edge f of F to $G[A]$ and $G[B]$ results in a connected graph and a spanning tree of this graph corresponds to a base of $\mathcal{M}(G)$. In particular, $F - f$ is contained in a complement of a base of $\mathcal{M}(G)$ for every $f \in F$ and thus the set $F - f$ is independent. We conclude that F is a circuit of $\mathcal{M}(G)$.

Let us now prove the other direction. Let F be a circuit of $\mathcal{M}^*(G)$. Since F is circuit of $\mathcal{M}^*(G)$, the graph $G \setminus F$ cannot be connected: otherwise, there would be a spanning tree of G avoiding all edges of F and F would be independent in $\mathcal{M}^*(G)$. Hence, F contains an edge-cut. Since every inclusion-wise minimal edge-cut is dependent (as we have already proven), F must be an inclusion-wise minimal edge-cut of G since F is an inclusion-wise minimal dependent set of $\mathcal{M}^*(G)$. □

We now show that in general, it is not true that the class of graphic matroids is closed under taking duals, i.e., there are graphic matroids whose dual is not graphic. This leads us to the definition of a cographic matroid: a matroid $\mathcal{M}$ is *cographic* if $\mathcal{M}$ is the dual of a graphic matroid.

Proposition 2.24. *Neither the matroid $\mathcal{M}^*(K_5)$ nor the matroid $\mathcal{M}^*(K_{3,3})$ is graphic.*

Proof. Let $\mathcal{M}^*(K_5)$ be isomorphic to a graphic matroid $\mathcal{M}(G)$ for a graph G. We may assume that G is connected (identifying vertices of different components will not change the structure of the cycles in G). Since $\mathcal{M}(K_5)$ has 10 elements and rank 4, G has 7 vertices and 10 edges and the average degree of G is less than 3. Let v be a vertex of degree at most 2 in G. The edges incident with v form an edge-cut of size at most two in G. It follows from Lemma 2.23 that $\mathcal{M}^*(G) = \mathcal{M}(K_5)$ has circuit of size 1 or 2 which is impossible. Hence, $\mathcal{M}^*(K_5)$ cannot be a graphic matroid.

Similarly, if $\mathcal{M}^*(K_{3,3})$ is isomorphic to a graphic matroid $\mathcal{M}(G)$, then G has 5 vertices and 9 edges. In particular, G has a vertex of degree at most three and $\mathcal{M}(K_{3,3})$ must contain a circuit formed by at most three elements. $\square$