Parameterized Inapproximability of Independent Set in H-Free Graphs

Pavel Dvořák[1⋆], Andreas Emil Feldmann[1⋆,⋆⋆], Ashutosh Rai[1⋆⋆], and Paweł Rzążewski[2,3⋆⋆⋆]

[1] Faculty of Mathematics and Physics, Charles University, Prague, Czechia
koblich@iuuk.mff.cuni.cz, feldmann.a.e@gmail.com, ashu.rai87@gmail.com

[2] Faculty of Mathematics and Information Science, Warsaw University of Technology, Warsaw, Poland
p.rzazewski@mini.pw.edu.pl

[3] University of Warsaw, Institute of Informatics, Warsaw, Poland

Abstract. We study the Independent Set problem in H-free graphs, i.e., graphs excluding some fixed graph H as an induced subgraph. We prove several inapproximability results both for polynomial-time and parameterized algorithms.

Halldórsson [SODA 1995] showed that for every $\delta > 0$ the Independent Set problem has a polynomial-time $(\frac{d-1}{2}+\delta)$-approximation algorithm in $K_{1,d}$-free graphs. We extend this result by showing that $K_{a,b}$-free graphs admit a polynomial-time $\mathcal{O}(\alpha(G)^{1-1/a})$-approximation, where $\alpha(G)$ is the size of a maximum independent set in G. Furthermore, we complement the result of Halldórsson by showing that for some $\gamma = \Theta(d/\log d)$, there is no polynomial-time γ-approximation algorithm for these graphs, unless NP=ZPP.

Bonnet *et al.* [IPEC 2018] showed that Independent Set parameterized by the size k of the independent set is W[1]-hard on graphs which do not contain (1) a cycle of constant length at least 4, (2) the star $K_{1,4}$, and (3) any tree with two vertices of degree at least 3 at constant distance. We strengthen this result by proving three inapproximability results under different complexity assumptions for almost the same class of graphs (we weaken condition (2) that G does not contain $K_{1,5}$). First, under the ETH, there is no $f(k) \cdot n^{o(k/\log k)}$ algorithm for any computable function f. Then, under the deterministic Gap-ETH, there is a constant $\delta > 0$ such that no δ-approximation can be computed in $f(k) \cdot n^{O(1)}$ time. Also, under the stronger randomized Gap-ETH there is no such approximation algorithm with runtime $f(k) \cdot n^{o(k)}$.

Finally, we consider the parameterization by the excluded graph H, and show that under the ETH, Independent Set has no $n^{o(\alpha(H))}$ algorithm in H-free graphs. Also, we prove that there is no $d/k^{o(1)}$-approximation algorithm for $K_{1,d}$-free graphs with runtime $f(d,k) \cdot n^{O(1)}$, under the deterministic Gap-ETH.

⋆ Supported by Czech Science Foundation GAČR (grant #19-27871X).

⋆⋆ Supported by Center for Foundations of Modern Computer Science (Charles Univ. project UNCE/SCI/004).

⋆⋆⋆ Supported by Polish National Science Centre grant no. 2018/31/D/ST6/00062.

1 Introduction

The Independent Set problem, which asks for a maximum sized set of pairwise non-adjacent vertices in a graph, is one of the most well-studied problems in algorithmic graph theory. It was among the first 21 problems that were proven to be NP-hard by Karp [22], and is also known to be hopelessly difficult to approximate in polynomial time: Håstad [21] proved that under standard assumptions from classical complexity theory the problem admits no $(n^{1-\varepsilon})$-approximation, for any $\varepsilon > 0$ (by n we always denote the number of vertices in the input graph). This was later strengthened by Khot and Ponnuswami [23], who were able to exclude any algorithm with approximation ratio $n/(\log n)^{3/4+\varepsilon}$, for any $\varepsilon > 0$. Let us point out that the currently best polynomial-time approximation algorithm for Independent Set achieves the approximation ratio $\mathcal{O}(n\frac{(\log\log n)^2}{(\log n)^3})$ [17].

There are many possible ways of approaching such a difficult problem, in order to obtain some positive results. One could give up on generality, and ask for the complexity of the problem on restricted instances. For example, while the Independent Set problem remains NP-hard in subcubic graphs [18], a straightforward greedy algorithm gives a 3-approximation.

H-free graphs. A large family of restricted instances, for which the Independent Set problem has been well-studied, comes from forbidding certain induced subgraphs. For a (possibly infinite) family $\mathcal{H}$ of graphs, a graph G is *$\mathcal{H}$-free* if it does not contain any graph of $\mathcal{H}$ as an induced subgraph. If $\mathcal{H}$ consists of just one graph, say $\mathcal{H} = \{H\}$, then we say that G is H-free. The investigation of the complexity of Independent Set in H-free graphs dates back to Alekseev, who proved the following.

Theorem 1 (Alekseev [2]). *Let $s \geq 3$ be a constant. The* Independent Set *problem is* NP*-hard in graphs that do not not contain any of the following induced subgraphs:*

1. *a cycle on at most s vertices,*
2. *the star $K_{1,4}$, and*
3. *any tree with two vertices of degree at least 3 at distance at most s.*

We can restate Theorem 1 as follows: the Independent Set problem is NP-hard in H-free graphs, unless H is a subgraph of a subdivided claw (i.e., three paths which meet at one of their endpoints). The reduction also implies that for each such H the problem is APX-hard and cannot be solved in subexponential time, unless the Exponential Time Hypothesis (ETH) fails. On the other hand, polynomial-time algorithms are known only for very few cases. First let us consider the case when $H = P_t$, i.e., we forbid a path on t vertices. Note that the case of $t = 3$ is trivial, as every P_3-free graph is a disjoint union of cliques. Already in 1981 Corneil, Lerchs, and Burlingham [11] showed that Independent Set is tractable for P_4-free graphs. For many years there was no improvement, until the breakthrough algorithm of Lokshtanov, Vatshelle, and Villanger [25] for P_5-free graphs. Their approach was recently extended to P_6-free graphs by

Grzesik, Klimošova, Pilipczuk, and Pilipczuk [19]. We still do not know whether the problem is polynomial-time solvable in P_7-free graphs, and we do not know it to be NP-hard in P_t-free graphs, for any constant t.

Even less is known for the case if H is a subdivided claw. The problem can be solved in polynomial time in claw-free (i.e., $K_{1,3}$-free) graphs, see Sbihi [32] and Minty [31]. This was later extended to H-free graphs, where H is a claw with one edge once subdivided (see Alekseev [1] for the unweighted version and Lozin, Milanič [27] for the weighted one).

When it comes to approximations, Halldórsson [20] gave an elegant local search algorithm that finds a $(\frac{d-1}{2} + \delta)$-approximation of the maximum independent set in $K_{1,d}$-free graphs for any constant $\delta > 0$ in polynomial time. Very recently, Chudnovsky, Thomassé, Pilipczuk, and Pilipczuk [10] designed a QPTAS (quasi-polynomial-time approximation scheme) that works for *every* H, which is a subgraph of a subdivided claw (in particular, a path). Recall that on all other graphs H the problem is APX-hard.

Parameterized complexity. Another approach that one could take is to look at the problem from the parameterized perspective: we no longer insist on finding the maximum independent set, but want to verify whether some independent set of size at least k exists. To be more precise, we are interested in knowing how the complexity of the problem depends on k. The best type of behavior we are hoping for is *fixed-parameter tractability* (FPT), i.e., the existence of an algorithm with running time $f(k) \cdot n^{\mathcal{O}(1)}$, for some function f (note that since the problem is NP-hard, we expect f to be super-polynomial).

It is known [12] that on general graphs the Independent Set problem is W[1]-hard parameterized by k, which is a strong indication that it does not admit an FPT algorithm. Furthermore, it is even unlikely to admit any non-trivial *fixed-parameter approximation (FPA)*: a γ-FPA algorithm for the Independent Set problem is an algorithm that takes as input a graph G and an integer k, and in time $f(k) \cdot n^{\mathcal{O}(1)}$ either correctly concludes that G has no independent set of size at least k, or outputs an independent set of size at least k/γ (note that γ does not have to be a constant). It was shown in [6] that on general graphs no $o(k)$-FPA exists for Independent Set, unless the randomized Gap-ETH fails.

Parameterized complexity in H-free graphs. As we pointed out, none of the discussed approaches, i.e., considering H-free graphs or considering parameterized algorithms, seems to make the Independent Set problem more tractable. However, some positive results can be obtained by combining these two settings, i.e., considering the parameterized complexity of Independent Set in H-free graphs. For example, the Ramsey theorem implies that any graph with $\Omega(4^p)$ vertices contains a clique or an independent set of size $\Omega(p)$. Since the proof actually tells us how to construct a clique or an independent set in polynomial time [16], we immediately obtain a very simple FPT algorithm for K_p-free graphs. Dabrowski [13] provided some positive and negative results for the complexity of the Independent Set problem in H-free graphs, for various H. The systematic study of the problem was initiated by Bonnet, Bousquet, Charbit, Thomassé, and Watrigant [4] and continued by Bonnet, Bousquet, Thomassé, and Watrigant [5]. Among other results, Bonnet *et al.* [4] obtained the following analog of Theorem 1.

Theorem 2 (Bonnet *et al.* [4]). *Let $s \geq 3$ be a constant. The* Independent Set *problem is* W[1]*-hard in graphs that do not contain any of the following induced subgraphs:*

1. *a cycle on at least 4 and at most s vertices,*
2. *the star $K_{1,4}$, and*
3. *any tree with two vertices of degree at least 3 at distance at most s.*

Note that, unlike in Theorem 1, we are not able to show hardness for C_3-free graphs: as already mentioned, the Ramsey theorem implies that Independent Set is FPT in C_3-free graphs. Thus, graphs H for which there is hope for FPT algorithms in H-free graphs are essentially obtained from paths and subdivided claws (or their subgraphs) by replacing each vertex with a clique.

Let us point out that, even though it is not stated there explicitly, the reduction of Bonnet *et al.* [4] also excludes any algorithm solving the problem in time $f(k) \cdot n^{o(\sqrt{k})}$, unless the ETH fails.

Our results. We study the approximation of the Independent Set problem in H-free graphs, mostly focusing on approximation hardness. Our first two results are related to Halldórsson's [20] polynomial-time $(\frac{d-1}{2}+\delta)$-approximation algorithm for $K_{1,d}$-free graphs. First, in Section 3 we extend this result to $K_{a,b}$-free graphs, for any constants a, b, showing the following theorem.

Theorem 3. *Given a $K_{a,b}$-free graph G, an $\mathcal{O}\big((a+b)^{1/a} \cdot \alpha(G)^{1-1/a}\big)$-approximation can be computed in $n^{\mathcal{O}(a)}$ time.*

Then, in Section 4 we show that the approximation ratio of the algorithm of Halldórsson [20] is optimal, up to logarithmic factors.

Theorem 4. *There is a function $\gamma = \Theta(d/\log d)$ such that the* Independent Set *problem does not admit a polynomial time γ-approximation algorithm in $K_{1,d}$-free graphs, unless* ZPP = NP.

Note that the factor γ determining the approximation gap in Theorem 4 is expressed as an asymptotic function of d, i.e., for growing d. In our case however, it is an interesting question how small the degree d can be so that we obtain an inapproximability result. We prove Theorem 4 by a reduction from the Label Cover problem, and a corresponding inapproximability result by Laekhanukit [24]. By calculating the bounds given in [24] (which heavily depend on the constant of Chernoff bounds) it can be shown that an inapproximability gap exists for $d \geq 31$ in Theorem 4.

Then in Section 5 we study the existence of fixed-parameter approximation algorithms for the Independent Set problem in H-free graphs. We show the following strengthening of Theorem 2, which also gives (almost) tight runtime lower bounds assuming the ETH or the randomized Gap-ETH (for more information about complexity assumptions used in Theorem 5 see Section 2).

Theorem 5. *Let $s \geq 4$ be a constant, and let $\mathcal{G}$ be the class of graphs that do not contain any of the following induced subgraphs:*

1. *a cycle on at least 5 and at most s vertices,*
2. *the star $K_{1,5}$, and*
3. *(i) the star $K_{1,4}$, or*
   (ii) a cycle on 4 vertices and any tree with two vertices of degree at least 3 at distance at most s.

The Independent Set *problem on $\mathcal{G}$ does not admit the following:*

(a) an exact algorithm with runtime $f(k) \cdot n^{o(k/\log k)}$, for any computable function f, under the ETH,

(b) a γ-approximation algorithm with runtime $f(k) \cdot n^{\mathcal{O}(1)}$ for some constant $\gamma > 0$ and any computable function f, under the deterministic Gap-ETH,

(c) a γ-approximation algorithm with runtime $f(k) \cdot n^{o(k)}$ for some constant $\gamma > 0$ and any computable function f, under the randomized Gap-ETH.

Finally, in Section 6 we study a slightly different setting, where the graph H is not considered to be fixed. As mentioned before, Independent Set is known to be polynomial-time solvable in P_t-free graphs for $t \leq 6$. The algorithms for increasing values of t get significantly more complicated and their complexity increases. Thus it is natural to ask whether this is an inherent property of the problem and can be formalized by a runtime lower bound when parameterized by t.

We give an affirmative answer to this question, even if the forbidden family is not a family of paths: note that the independent set number $\alpha(P_t)$ of a path on t vertices is $\lceil t/2 \rceil$.

Proposition 1. *Let d be an integer and let $\mathcal{H}_d$ be a family of graphs, such that $\alpha(H) > d$ for every $H \in \mathcal{H}_d$. The* Independent Set *problem in $\mathcal{H}_d$-free graphs is* W[1]*-hard parameterized by d and cannot be solved in $n^{o(d)}$ time, unless the ETH fails.*

We also study the special case when $H = K_{1,d}$ and consider the inapproximability of the problem parameterized by both $\alpha(K_{1,d}) = d$ and k. Unfortunately, for the parameterized version we do not obtain a clear-cut statement as in Theorem 4, since in the following theorem d cannot be chosen independently of k in order to obtain an inapproximability gap.

Proposition 2. *Let $\varepsilon > 0$ be any constant and $\xi(k) = 2^{(\log k)^{1/2+\varepsilon}}$. The* Independent Set *problem in $K_{1,d}$-free graphs has no $d/\xi(k)$-approximation algorithm with runtime $f(d,k) \cdot n^{\mathcal{O}(1)}$ for any computable function f, unless the deterministic Gap-ETH fails.*

Note that this in particular shows that if we allow d to grow as a polynomial k^ε for any constant $\frac{1}{2} > \varepsilon > 0$, then no k^δ-approximation is possible for any $\delta < \varepsilon$ (since $\xi(k) = k^{o(1)}$). This indicates that the $(\frac{d-1}{2} + \delta)$-approximation for $K_{1,d}$-free graphs [20] is likely to be best possible (up to sub-polynomial factors), even when parameterizing by k and d. The proofs of Proposition 1 and Proposition 2 can be found in Section 6.

2 Preliminaries

All our hardness results for INDEPENDENT SET are obtained by reductions from some variant of the MAXIMUM COLORED SUBGRAPH ISOMORPHISM (MCSI) problem. This optimization problem has been widely studied in the literature, both to obtain polynomial-time and parameterized inapproximability results, but also in its decision version to obtain parameterized runtime lower bounds. We note that by applying standard transformations, MCSI contains the well-known problems LABEL COVER [24] and BINARY CSP [26]: for BINARY CSP the graph J is a complete graph, while for LABEL COVER J is usually bipartite.

MAXIMUM COLORED SUBGRAPH ISOMORPHISM (MCSI)	
Input:	A graph G, whose vertex set is partitioned into subsets $V_1, \dots, V_\ell$, and a graph J on vertex set $\{1, \dots, \ell\}$.
Goal:	Find an assignment $\phi : V(J) \to V(G)$, where $\phi(i) \in V_i$ for every $i \in [\ell]$, that maximizes the number $S(\phi)$ of satisfied edges, i.e., $S(\phi) := \big|\{ij \in E(J) \mid \phi(i)\phi(j) \in E(G)\}\big|$.

Given an instance $\Gamma = (G, V_1, \dots, V_\ell, J)$ of MCSI, we refer to the number of vertices of G as the *size* of Γ. Any assignment $\phi : V(J) \to V(G)$, such that for every i it holds that $\phi(i) \in V_i$, is called a *solution of* Γ. The *value* of a solution ϕ is $\text{val}(\phi) := S(\phi)/|E(J)|$, i.e., the fraction of satisfied edges. The value of the instance Γ, denoted by $\text{val}(\Gamma)$, is the maximum value of any solution of Γ.

When considering the decision version of MCSI, i.e., determining whether $\text{val}(\Gamma) = 1$ or $\text{val}(\Gamma) < 1$, a result by Marx [29] gives a runtime lower bound for parameter ℓ under the *Exponential Time Hypothesis (ETH)*. That is, no $f(\ell) \cdot n^{o(\ell/\log \ell)}$ time algorithm can solve MCSI for any computable function f, assuming there is no deterministic $2^{o(n)}$ time algorithm to solve the 3-SAT problem. For the optimization version, an α-approximation is a solution ϕ with $\text{val}(\phi) \geq 1/\alpha$. When J is a complete graph, a result by Dinur and Manurangsi [14,15] states that there is no $\ell/\xi(\ell)$-approximation algorithm (where $\xi(\ell) = 2^{(\log \ell)^{1/2+\varepsilon}}$ for any constant $\varepsilon > 0$) with runtime $f(\ell) \cdot n^{O(1)}$ for any computable function f, unless the *deterministic Gap-ETH* fails (see Theorem 11). This hypothesis assumes that there exists some constant $\delta > 0$ such that no deterministic $2^{o(n)}$ time algorithm for 3-SAT can decide whether all or at most a $(1-\delta)$-fraction of the clauses can be satisfied. A recent result by Manurangsi [28] uses an even stronger assumption, which also rules out randomized algorithms, and in turn obtains a better runtime lower bound at the expense of a worse approximation lower bound: he shows that, when J is a complete graph, there is no γ-approximation algorithm for MCSI with runtime $f(\ell) \cdot n^{o(\ell)}$ for any computable function f and any constant γ, under the *randomized Gap-ETH*. This assumes that there exists some constant $\delta > 0$ such that no randomized $2^{o(n)}$ time algorithm for 3-SAT can decide whether all or at most a $(1-\delta)$-fraction of the clauses can be satisfied. (Note that the runtime lower bound under the stronger randomized Gap-ETH does not have the $\log(\ell)$ factor in the polynomial degree as the runtime lower bound under ETH does.)

For our results we will often need the special case of MCSI when the graph J has bounded degree. We define this problem in the following.

Degree-t Maximum Colored Subgraph Isomorphism (MCSI(t))
Input: A graph G, whose vertex set is partitioned into subsets $V_1, \dots, V_\ell$, and a graph J on vertex set $\{1, \dots, \ell\}$ and maximum degree t.
Goal: Find an assignment $\phi : V(J) \to V(G)$, where $\phi(i) \in V_i$ for every $i \in [\ell]$, that maximizes the number $S(\phi)$ of satisfied edges, i.e., $S(\phi) := \left|\{ij \in E(J) \mid \phi(i)\phi(j) \in E(G)\}\right|$.

The bounded degree case has been considered before, and we harness some of the known hardness results for MCSI(t) in our proofs. First, let us point out that the lower bound for exact algorithms holds even for the case when $t = 3$, as shown by Marx and Pilipczuk [30]. We also use a polynomial-time approximation lower bound given by Laekhanukit [24], where t can be set to any constant and the approximation gap depends on t (see Theorem 6). The complexity assumption of this algorithm is that NP-hard problems do not have polynomial time Las Vegas algorithms, i.e., NP $\neq$ ZPP. For parameterized approximations, we use a result by Lokshtanov et al. [26], who obtain a constant approximation gap for the case when $t = 3$ (see Theorem 8). It seems that this result for parameterized algorithms is not easily generalizable to arbitrary constants t so that the approximation gap would depend only on t, as in the result for polynomial-time algorithms provided by Laekhanukit [24]: neither the techniques found in [24] nor those of [26] seem to be usable to obtain an approximation gap that depends only on t but not the parameter ℓ. However, we develop a weaker parameterized inapproximability result for the case when $t \geq \xi(\ell) = \ell^{o(1)}$ (see Theorem 9 in Section 6), and use it to prove Proposition 2.

3 Approximation for $K_{a,b}$-free graphs

In this section we give a polynomial-time $\mathcal{O}\big((a+b)^{1/a} \cdot \alpha(G)^{1-1/a}\big)$-approximation algorithm for Independent Set on $K_{a,b}$-free graphs, where $\alpha(G)$ is the size of the maximum independent set in the input graph G. The algorithm is a generalization of a known local search procedure. Note that it asymptotically matches the approximation factor of the $(\frac{d-1}{2} + \delta)$-approximation algorithm for $K_{1,d}$-free graphs of Halldórsson [20] by setting $a = 1$ and $b = d$. We note here that the following theorem was independently discovered by Bonnet, Thomassé, Tran, and Watrigant [3].

Theorem 3. *Given a $K_{a,b}$-free graph G, an $\mathcal{O}\big((a+b)^{1/a} \cdot \alpha(G)^{1-1/a}\big)$-approximation can be computed in $n^{\mathcal{O}(a)}$ time.*

Proof. The algorithm first computes a maximal independent set $I \subseteq V(G)$ in the given graph G, which can be done in linear time using a simple greedy approach. Since I is maximal, every vertex in $V(G) \setminus I$ has at least one neighbor in I. Now, we consider the vertices in $V(G) \setminus I$ that are neighbors to at most $a-1$ vertices of I, and call this set V_1. Let $C \subseteq I$ be a set of size $c \in [a-1]$, and let $V_C := \{v \in V_1 \mid N(v) \cap I = C\}$. If the graph induced by $V_C \cup C$ contains an independent set I' of size $|C|+1$, then we can find it in time $n^{\mathcal{O}(|C|+1)} = n^{\mathcal{O}(a)}$. Furthermore, $(I \setminus C) \cup I'$ is an independent set, since no vertex of $V_C \cup C$ is adjacent to any vertex of $I \setminus C$, and $(I \setminus C) \cup I'$ is larger by one than I. Thus the algorithm replaces $I \setminus C$ by I' in I. The algorithm repeats this procedure until the largest independent set in each subgraph induced by a set $V_C \cup C$ (defined for the current I) is of size at most $|C|$. At this point the algorithm outputs I.

Let $k = |I|$ be the size of the output at the end of the algorithm. We claim that $\alpha(G) \leq (a-1)(ek)^{a-1} + (b-1)k^a = \mathcal{O}\big((a+b)k^a\big)$ and this would prove the theorem, since then $k = \Omega\big((\frac{\alpha(G)}{a+b})^{1/a}\big)$, which implies that I is an $\mathcal{O}\big((a+b)^{1/a} \cdot \alpha(G)^{1-1/a}\big)$-approximation.

To show the claim, first note that the family $\big\{V_C \mid C \subseteq I \text{ and } |C| \in [a-1]\big\}$ is a partition V_1 into at most $\sum_{c=1}^{a-1} \binom{k}{c}$ many sets. For each relevant C, no subgraph induced by a set $V_C \cup C$ contains an independent set larger than $|C|$, and thus if I^* denotes the maximum independent set of G, then $\big|(V_C \cup C) \cap I^*\big| \leq |C|$. Thus,

$$\big|(V_1 \cup I) \cap I^*\big| \leq \sum_{c=1}^{a-1} c\binom{k}{c} \leq \sum_{c=1}^{a-1} c(ek/c)^c \leq (a-1)(ek)^{a-1}.$$

Now consider the remaining set $V_2 := V(G) \setminus (V_1 \cup I)$, and observe that every $v \in V_2$ has at least a neighbors in I due to the definition of V_1. For each $D \subseteq I$ with $|D| = a$, we construct a set V_D by fixing an arbitrary subset $S(v) \subseteq (N(v) \cap I)$ of size a for every $v \in V_2$, and putting v into V_D if and only if $S(v) = D$. Observe that these sets V_D form a partition of V_2 of size at most $\binom{k}{a}$. We claim that each V_D induces a subgraph of G for which every independent set has size less than b. Assume not, and let I' be an independent set in V_D of size b. But then $D \cup I'$ induces a $K_{a,b}$ in G, since every vertex of $I' \subseteq V_D$ is adjacent to every vertex of $D \subseteq I$. As this contradicts the fact that G is $K_{a,b}$-free, we have $|V_D \cap I^*| \leq b-1$, and consequently $|V_2 \cap I^*| \leq (b-1)\binom{k}{a} \leq (b-1)k^a$. Together with the above bound on the number of vertices of I^* in $V_1 \cup I$ we get

$$\alpha(G) = |I^*| \leq (a-1)(ek)^{a-1} + (b-1)k^a,$$

which concludes the proof. □

4 Polynomial time inapproximability in $K_{1,d}$-free graphs

In this section, we show polynomial time approximation lower bounds for Independent Set on $K_{1,d}$-free graphs.

Theorem 4. *There is a function $\gamma = \Theta(d/\log d)$ such that the* Independent Set *problem does not admit a polynomial time γ-approximation algorithm in $K_{1,d}$-free graphs, unless* ZPP = NP.

For that, we reduce from the MCSI(t) problem, and leverage the lower bound by Laekhanukit [24, Theorem 6]. Let us point out that the original statement of the lower bound by Laekhanukit [24] is in terms of the Label Cover problem, but, as we already mentioned, it is equivalent to MCSI.

Theorem 6 (Laekhanukit [24]). *Let $\Gamma = (G, V_1, \ldots, V_\ell, J)$ be an instance of* MCSI(t) *where J is a bipartite graph. Assuming* ZPP $\neq$ NP*, there exists a constant c such that for any constant $\varepsilon > 0$, there is no polynomial time algorithm that can distinguish between the two cases:*
1. *(YES-case)* $\mathrm{val}(\Gamma) \geq 1 - \varepsilon$, *and*
2. *(NO-case)* $\mathrm{val}(\Gamma) \leq c \log(t)/t + \varepsilon$.

We use a standard reduction from MCSI to Independent Set, which for instances of MCSI(t) of bounded degree t gives the following lemma.

Lemma 1. *Let $\Gamma = (G, V_1, \ldots, V_\ell, J)$ be an instance of* MCSI(t). *Given Γ, in polynomial time we can construct an instance G' of* Independent Set *such that*
1. *G' does not have $K_{1,d}$ as an induced subgraph for any $d \geq 2t + 2$,*
2. *if $\mathrm{val}(\Gamma) \geq \mu$ then G' has an independent set of size at least $\mu|E(J)|$, and*
3. *if $\mathrm{val}(\Gamma) \leq \nu$ then every independent set of G' has size at most $\nu|E(J)|$.*

Proof. We first describe the construction of G' given $\Gamma = (G, V_1, \ldots, V_\ell, J)$, where we denote by E_{ij} the edge set between V_i and V_j for each edge $ij \in E(J)$. The graph G' has a vertex v_e for each edge e of G, an edge between v_e and v_f if $e, f \in E_{ij}$ for some $ij \in E(J)$, and an edge between v_e and v_f if $e \in E_{ij}$ and $f \in E_{ij'}$ and e and f do not share a vertex in G for some three vertices $i, j, j' \in [\ell]$ of J such that $ij \in E(J)$ and $ij' \in E(J)$. Note that the vertex set $V'_{ij} = \{v_e \in V(G') \mid e \in E_{ij}\}$ induces a clique in G'. This finishes the construction of G'. See Figure 1 for better understanding of the construction.

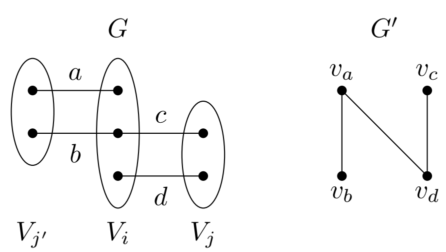

Fig. 1. Sketch of the construction of the graph G'.

To see the first part of the lemma, for the sake of contradiction, let us suppose G' has a $K_{1,d}$ as an induced subgraph for $d \geq 2t+2$. We know that for any $e \in E(J)$ the vertices in V'_e form a clique in G', so the star $K_{1,d}$ can intersect with a fixed V'_e in at most two vertices of which one must be the center vertex of

$K_{1,d}$ with degree d. As $K_{1,d}$ has $d+1$ vertices, this means there are (at least) d distinct vertex sets $V'_{e_1}, V'_{e_2}, \ldots, V'_{e_d}$ of G that intersect the $K_{1,d}$ for some edges $e_1, e_2, \ldots, e_d \in E(J)$. Without loss of generality, let the center vertex of the $K_{1,d}$ come from V'_{e_1}. Note that the $K_{1,d}$ has an edge between a vertex from V'_{e_1} and a vertex from V'_{e_i} for each $i \in \{2, \ldots, d\}$. Hence if $e_1 = jj'$, we have that either $j \in e_i$ or $j' \in e_i$ for every $i \in [d]$ by the construction of G'. This means that either j or j' has at least $(d-1)/2$ neighbours in J. That is, $\Delta(J) \geq (d-1)/2$, which gives a contradiction to $\Delta(J) \leq t$ and $d \geq 2t+2$.

Now, to see the second claim of the lemma, first we need to show that if $\mathrm{val}(\Gamma) \geq \mu$, then G' has an independent set of size at least $\mu|E(J)|$. To see that, let $\phi : V(J) \to V(G)$ be a mapping that satisfies at least a μ-fraction of the edges of $E(J)$. We claim that $S = \{v_{uw} \in V'_{ij} \mid ij \in E(J), \phi(i) = u, \phi(j) = w\}$ is an independent set of size at least $\mu|E(J)|$ in G'. Since ϕ satisfies at least μ-fraction of edges, S has size at least $\mu|E(J)|$. So all we need to show is that S is indeed an independent set. Suppose it was not the case, i.e., there exist $v_e, v_f \in S$ that are adjacent in G'. By construction of G' there can be an edge between v_e and v_f only if $e \in E_{ij}$ and $f \in E_{ij'}$ where possibly $j = j'$. Note that $\phi(i) = u \in V_i$ is a common endpoint of both e and f. If indeed $j = j'$, then $\phi(j) = w \in V_j$ is also a common endpoint of both e and f, so that $e = f$, i.e., v_e and v_f are not distinct. Hence it must be that $j \neq j'$. But in this case, the construction of G' implies that e and f do not share a vertex, which contradicts the fact that they have u as a common endpoint.

For the third part of the lemma, we prove the contrapositive: we claim that if G' has an independent set S of size $k \geq \nu|E(J)|$, then there exists an assignment $\phi : V(J) \to V(G)$ satisfying at least k edges in Γ. To see that, first observe that the set S can contain at most one vertex from V'_e as any two vertices in V'_e are adjacent. Let $E_S := \{e \in E(J) \mid S \cap V'_e \neq \emptyset\}$, for which we then have $|E_S| = |S|$. We claim that all the edges in E_S can be satisfied by an assignment ϕ defined as follows. For $ij \in E_S$, let $S \cap V'_{ij} = \{v_{uw}\}$. Then we set $\phi(i) = u$ and $\phi(j) = w$. We need to show that the function ϕ is well-defined. Suppose some vertex $i \in V(J)$ gets mapped to more than one vertex of $V(G)$ by ϕ. This must mean that there exist two edges in G that contain one endpoint in V_i and are in E_S. But this would mean that the two vertices in S corresponding to these two edges in E_S are adjacent due to the construction of G'. This is a contradiction to S being an independent set. Also, $\phi(i)\phi(j) \in E(G)$ for all $ij \in E_S$, since for each v_{uw} we have $uw \in E(G)$, and we have set $\phi(i) = u$ and $\phi(j) = w$. This concludes the proof. □

Now we are ready to prove Theorem 4.

Proof of Theorem 4. Assume there was a polynomial time algorithm $\mathcal{A}$ to approximate the Independent Set problem within a factor $\frac{1-\varepsilon}{c\log(t)/t+\varepsilon}$ for some $\varepsilon > 0$ in $K_{1,d}$-free graphs, where $t = \lfloor \frac{d}{2} - 1 \rfloor$, and c is the constant given by Theorem 6. Given an instance $\Gamma = (G, V_1, \ldots, V_\ell, J)$ of MCSI(t) and ε, we can reduce it to an instance of Independent Set in $K_{1,d}$-free graphs in polynomial time by using the reduction of Lemma 1. Now, setting $\mu = 1 - \varepsilon$

and $\nu = (c \log(t)/t) + \varepsilon$ in the statement of Lemma 1, this gives that given an instance Γ of MCSI(t) and ε, we can now use $\mathcal{A}$ to differentiate between the YES- and NO-cases of Theorem 6 in polynomial time, which would mean that $\mathsf{ZPP} = \mathsf{NP}$. As $\frac{1-\varepsilon}{c\log(t)/t+\varepsilon} = \mathcal{O}(d/\log d)$, this implies Theorem 4. □

5 Parameterized approximation for fixed H

In this section we prove Theorem 5. Let us define an auxiliary family of classes of graphs: for integers $4 \leq a \leq b$ and $c \geq 3$, let $\mathcal{C}([a,b],c)$ denote the class of graphs that are $K_{1,c}$-free and C_p-free for any $p \in [a,b]$. Let $\mathcal{T}(b)$ be the class of trees with two vertices of degree at least 3 at distance at most b. Let $\mathcal{C}^*([a,b],c) \subseteq \mathcal{C}([a,b],c)$ be the set of those $G \in \mathcal{C}([a,b],c)$, which are are also $\mathcal{T}(\lceil \frac{b-1}{2} \rceil)$-free. Actually, we will prove the following theorem, which implies Theorem 5.

Theorem 7. *Let $z \geq 5$ be a constant. The following lower bounds hold for the* Independent Set *problem on graphs $G \in \mathcal{C}^*([4,z],5) \cup \mathcal{C}([5,z],4)$ with n vertices.*

1. *For any computable function f, there is no $f(k) \cdot n^{o(k/\log k)}$-time algorithm that determines if $\alpha(G) \geq k$, unless the ETH fails.*
2. *There exists a constant $\gamma > 0$, such that for any computable function f, there is no $f(k) \cdot n^{\mathcal{O}(1)}$-time algorithm that can distinguish between the two cases: $\alpha(G) \geq k$, or $\alpha(G) < (1-\gamma) \cdot k$, unless the deterministic Gap-ETH fails.*
3. *There exists a constant $\gamma > 0$, such that for any computable function f, there is no $f(k) \cdot n^{o(k)}$-time algorithm that can distinguish between the two cases: $\alpha(G) \geq k$, or $\alpha(G) < (1-\gamma) \cdot k$, unless the randomized Gap-ETH fails.*

The proof of Theorem 7 consists of two steps: first we will prove it for graphs in $\mathcal{C}^*([4,z],5)$, and then for graphs in $\mathcal{C}([5,z],4)$. In both proofs we will reduce from the MCSI(3) problem. Let $\Gamma = (G, V_1, \ldots, V_\ell, H)$ be an instance of MCSI(3). For $ij \in E(H)$, by $E_{ij} = E_{ji}$ we denote the set of edges between V_i and V_j. Note that we may assume that H has no isolated vertices, each V_i is an independent set, and $E_{ij} \neq \emptyset$ if and only if $ij \in E(H)$.

Lokshtanov et al. [26] gave the following hardness result (the first statement actually follows from Marx [29] and Marx, Pilipczuk [30]). We note that Lokshtanov et al. [26] conditioned their result on the *Parameterized Inapproximability Hypothesis* (PIH) and $\mathsf{W[1]} \neq$ FPT. Here we use stronger assumptions, i.e., the deterministic and randomized Gap-ETH, which are more standard in the area of parameterized approximation. The reduction in [26] yields the following theorem, when starting from [14,15] and [28], respectively (see also [8, Corollary 7.9]).

Theorem 8 (Lokshtanov et al. [26]). *Consider an arbitrary instance $\Gamma = (G, V_1, \ldots, V_\ell, H)$ of* MCSI(3) *with size n.*

1. *Assuming the ETH, for any computable function f, there is no $f(\ell) \cdot n^{o(\ell/\log \ell)}$ time algorithm that solves Γ.*
2. *Assuming the deterministic Gap-ETH there exists a constant $\gamma > 0$, such that for any computable function f, there is no $f(\ell) \cdot n^{\mathcal{O}(1)}$ time algorithm*

that can distinguish between the two cases: (YES-case) $\mathrm{val}(\Gamma) = 1$*, and* (NO-case) $\mathrm{val}(\Gamma) < 1 - \gamma$*.*

3. *Assuming the randomized Gap-ETH there exists a constant $\gamma > 0$, such that for any computable function f, there is no $f(\ell) \cdot n^{o(\ell)}$ time algorithm that can distinguish between the two cases:* (YES-case) $\mathrm{val}(\Gamma) = 1$*, and* (NO-case) $\mathrm{val}(\Gamma) < 1 - \gamma$*.*

5.1 Hardness for $(C_4, C_5 \ldots, C_z, K_{1,5}, \mathcal{T}(\lceil \frac{z-1}{2} \rceil))$-free graphs

First, let us show Theorem 7 for $\mathcal{C}^*([4, z], 5)$, i.e., for $(C_4, C_5 \ldots, C_z, K_{1,5}, \mathcal{T}(s))$-free graphs for $s = \lceil \frac{z-1}{2} \rceil$. Let $\Gamma = (G, V_1, \ldots, V_\ell, H)$ be an instance of MCSI(3). We aim to build an instance (G', k) of Independent Set, such that the graph $G' \in \mathcal{C}^*([4, z], 5)$.

For each $ij \in E(H)$, we introduce a clique C_{ij} of size $|E_{ij}|$, whose every vertex *represents* a different edge from E_{ij}. The cliques constructed at this step will be called *primary cliques*, note that their number is $|E(H)|$. Choosing a vertex v from C_{ij} to an independent set of G' will correspond to mapping i and j to the appropriate endvertices of the edge from E_{ij}, corresponding to v.

Now we need to ensure that the choices in primary cliques corresponding to edges of G are consistent. Consider $i \in V(H)$ and suppose it has three neighbors j_1, j_2, j_3 (the cases if i has fewer neighbors are dealt with analogously). We will connect the cliques $C_{ij_1}, C_{ij_2}, C_{ij_3}$ using a gadget called a *vertex-cycle*, whose construction we describe below. For each $a \in \{1, 2, 3\}$, we introduce s copies of C_{ij_a} and denote them by $D^1_{ij_a}, D^2_{ij_a}, \ldots, D^s_{ij_a}$, respectively. Let us call these copies *secondary cliques*. The vertices of secondary cliques represent the edges from E_{ij_a} analogously as the ones of C_{ij_a}. We call primary and secondary cliques as *base cliques*. We connect the base cliques corresponding to the vertex $i \in V(H)$ into *vertex-cycle* $\mathcal{C}_i$. Imagine that secondary cliques, along with primary cliques $C_{ij_1}, C_{ij_2}, C_{ij_3}$, are arranged in a cycle-like fashion, as follows:

$$C_{ij_1}, D^1_{ij_1}, D^2_{ij_1}, \ldots, D^s_{ij_1}, C_{ij_2}, D^1_{ij_2}, D^2_{ij_2}, \ldots, D^s_{ij_2}, C_{ij_3}, D^1_{ij_3}, D^2_{ij_3}, \ldots, D^s_{ij_3}, C_{ij_1}.$$

This cyclic ordering of cliques constitutes the vertex-cycle, let us point out that we treat this cycle as a directed one. As we describe below we put some edges between two base cliques D_1 and D_2 only if they belong to some vertex-cycle $\mathcal{C}_i$. See Figure 2 for an example of how we connect base cliques.

Now, we describe how we connect the consecutive cliques in $\mathcal{C}_i$. Recall that each vertex v of each clique represents exactly one edge uw of G, whose exactly one vertex, say u, is in V_i. We extend the notion of representing and say that v *represents* u, and denote it by $r_i(v) = u$.

Let us fix an arbitrary ordering $\prec_i$ on V_i. Now, consider two consecutive cliques of the vertex-cycle. Let v be a vertex of the first clique and v' be a vertex from the second clique, and let u and u' be the vertices of V_i represented by v and v', respectively. The edge vv' exists in G' if and only if $u \prec_i u'$. See Figure 3 how we connect two consecutive base cliques in a vertex-cycle. This finishes the construction of $\mathcal{C}_i$. We introduce a vertex-cycle $\mathcal{C}_i$ for every vertex i of H, note that each primary clique C_{ij} is in exactly two vertex-cycles: $\mathcal{C}_i$ and $\mathcal{C}_j$. The number of all base cliques is

$$k := \underbrace{|E(H)|}_{\substack{\text{primary}\\ \text{cliques}}} + \underbrace{\sum\nolimits_{i \in V(H)} \deg_H(i) \cdot s}_{\text{secondary cliques}} = |E(H)| \cdot \left(1 + \frac{s}{2}\right) \leq \frac{3\ell}{2} \cdot \left(1 + \frac{s}{2}\right) = \mathcal{O}(\ell).$$

This concludes the construction of (G', k). Since $V(G')$ is partitioned into k base cliques, k is an upper bound on the size of any independent set in G', and a solution of size k contains exactly one vertex from each base clique.

We claim that the graph G' is in the class $\mathcal{C}^*([4, z], 5)$. Moreover, if $\mathrm{val}(\Gamma) = 1$, then the graph G' has an independent set of size k and if the graph G' has an independent set of size at least $(1 - \gamma') \cdot k$ for $\gamma' = \frac{\gamma}{6+3s}$, then $\mathrm{val}(\Gamma) \geq 1 - \gamma$.

For two distinct base cliques D_1, D_2, by $E(D_1, D_2)$ we denote the set of edges with one endvertex in D_1 and another in D_2. We say that D_1, D_2 are *adjacent* if $E(D_1, D_2) \neq \emptyset$.

Claim 5.1. *Let D_1, D_2 be two distinct base cliques in G'. Then the size of a maximum induced matching in the graph induced by $E(D_1, D_2)$ is at most 1.*

Proof. If $E(D_1, D_2)$ is empty, then the lemma holds trivially. Consider two disjoint edges $e_1 = v_1w_1$ and $e_2 = v_2w_2$ in $E(D_1, D_2)$, where $v_1, v_2 \in D_1$ and $w_1, w_2 \in D_2$. We prove that there is an edge $e \in E(D_1, D_2)$ such that e intersect both e_1 and e_2.

By construction, D_1 and D_2 are consecutive cliques in a vertex-cycle $\mathcal{C}_i$ for some $i \in V(H)$. Assume that D_2 is the successor of D_1 on this cycle. Recall that each $u \in \{v_1, v_2, w_1, w_2\}$ represents some vertex $r_i(u) \in V_i$. Since $v_1w_1, v_2w_2 \in E(G')$, we observe that $r_i(v_1) \prec_i r_i(w_1)$ and $r_i(v_2) \prec_i r_i(w_2)$. Thus, either $r_i(v_1) \prec_i r_i(w_2)$ or $r_i(v_2) \prec_i r_i(w_1)$, so one of the edges v_1w_2 or v_2w_1 exists in G'. □

Claim 5.2. *The graph G' is $(C_4, \ldots, C_z)$-free.*

Proof. For contradiction, suppose that there exists an induced cycle K in G' with consecutive vertices $(v_1, v_2, \ldots, v_p)$, where $p \in [4, s]$. Note that two consecutive

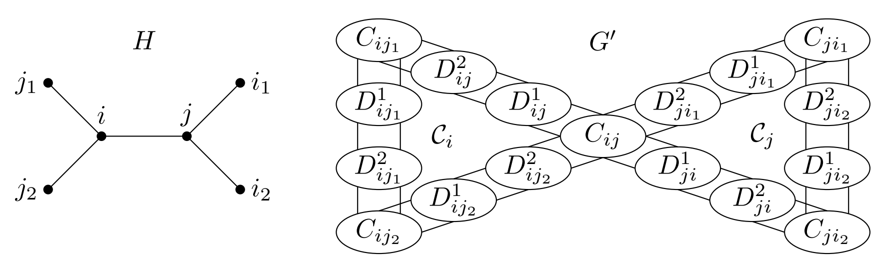

Fig. 2. A part of the construction of G' for $s = 2$. Cliques C_{ab} representing edge sets $E_{ab} \subseteq E(G)$ are connected through secondary cliques D^p_{ab}.

vertices of K might be in the same base clique, or two adjacent base cliques. Furthermore, no non-consecutive vertices of K may be in one base clique.

Note that each vertex-cycle in G' has at least $2s + 2 > z$ base cliques, so K cannot intersect more than two base cliques. It cannot intersect one base clique, as $p > 3$, so suppose that K intersects exactly two base cliques D_1 and D_2. Observe that this means that $p = 4$ and $v_1, v_2 \in D_1$, while $v_3, v_4 \in D_2$. However, by Claim 5.1, we observe that either v_1 and v_3, or v_2 and v_4, are adjacent in G', so K is not induced. □

Claim 5.3. *The graph G' is $K_{1,5}$-free.*

Proof. By contradiction suppose that the set $\{v, u_1, u_2, u_3, u_4, u_5\} \subseteq V(G')$ induces a copy of $K_{1,5}$ in G' with v being the central vertex. Let C_{ij} be the base clique containing v. Since each of u_1, u_2, u_3, u_4, u_5 must be in a different base clique and C_{ij} is adjacent to at most four other base cliques, we conclude that one of u_i's, say u_5, belongs to C_{ij}. For $i \in [4]$, let D_i be the base clique containing u_i. Furthermore, note that C_{ij} must be a primary clique, since only those ones are adjacent to four base cliques. Therefore two of D_i's, say D_1 and D_2, must belong to the vertex-cycle $\mathcal{C}_i$. Let u_1 precede v, and u_2 succeed v on this cycle. Consider the vertices $r_i(v), r_i(u_1), r_i(u_2), r_i(u_5)$ and recall that since v is adjacent to u_1, u_2, we have $r_i(u_1) \prec_i r_i(v) \prec_i r_i(u_2)$. However, u_5 is non-adjacent to u_1, u_2, which means that $r_i(u_2) \prec_i r_i(u_5) \prec_i r_i(u_1)$, which is a contradiction, since $\prec_i$ is transitive. □

Claim 5.4. *Let $T \in \mathcal{T}(s)$. Then, the graph G' is T-free.*

Proof. Suppose that G' contains T as an induced subgraph. Let $u, v \in V(T)$ such that $\deg_T(u), \deg_T(v) \geq 3$ and $\mathrm{dist}_T(u, v) \leq s$. Note that primary cliques are at distance $s+1$. Thus, u and v can not be both in primary cliques. Without loss of generality, let v be in a secondary clique D of a vertex-cycle $\mathcal{C}_i$. There are only two base cliques D_1 and D_2 adjacent to the secondary clique D. Let u_1, u_2 and u_3 be neighbors of v in T. Since u_1, u_2 and u_3 form an independent set in T they have to be in different base cliques in G. Thus, we can suppose $u_1 \in V(D_1), u_2 \in V(D_2)$ and $u_3 \in V(D)$. However, by the same argument as in proof of Claim 5.3 these four vertices v, u_1, u_2 and u_3 can not exist. □

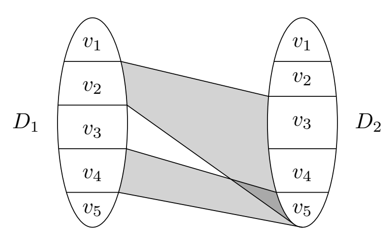

Fig. 3. Edges between two consecutive cliques D_1 and D_2 in a vertex-cycle $\mathcal{C}_i$, where $V_i = \{v_1, \ldots, v_5\}$. We show only edges incident to $u \in V(D_1)$ such that $r_i(u) \in \{v_2, v_4\}$.

Claim 5.5. *If* $\mathrm{val}(\Gamma) = 1$*, then the graph* G' *has an independent set of size* k*.*

Proof. Let ϕ be a solution of Γ of value 1, i.e., for each $ij \in E(H)$ holds that $\phi(i)\phi(j)$ is an edge of G. We will find an independent set I in G' of size k. For each $ij \in E(H)$ we add to the set I a vertex from the primary clique C_{ij} which represents the edge $\phi(i)\phi(j)$. Thus, we pick one vertex from each primary clique. Recall that each secondary clique D is a copy of some primary clique C. If we pick a vertex v from C then we add to I also a copy of v from D. Thus, we add one vertex from each base clique to the set I and therefore $|I| = k$.

We claim that I is independent. Suppose there exist $v, w \in I$ such that $vw \in E(G')$. Let $v \in V(D_1)$ and $w \in V(D_2)$ for some base cliques D_1 and D_2. First, suppose that D_1 and D_2 are copies of the same primary clique C_{ij} (or one of them is the primary clique itself and the second one is the copy)[4]. Thus, the vertices v and w represent the same edge in E_{ij} and by construction, vertices in primary and secondary cliques representing the same edge in E_{ij} are not adjacent.

Therefore $D_1 = D^s_{ij_1}$ and $D_2 = C_{ij_2}$ (or vice versa) for some edges ij_1 and ij_2 in $E(H)$. Edges between D_1 and D_2 were added according to the ordering $\prec_i$ of vertices in V_i. Note that the vertices v and w represent edges $\phi(i)\phi(j_1)$ and $\phi(i)\phi(j_2)$. Thus, $r_i(v) = \phi(i) = r_i(w)$. Since v and w are adjacent in G', it holds that $r_i(v) \prec_i r_i(w)$ by construction, which is a contradiction with $r_i(v) = r_i(w)$. Therefore, I is an independent set. □

Claim 5.6. *Let* $\gamma > 0$*. If the graph* G' *has an independent set of size at least* $(1 - \gamma') \cdot k$ *for* $\gamma' = \frac{\gamma}{6+3s}$*, then* $val(\Gamma) \geq 1 - \gamma$*.*

Proof. Let
- I be a maximum independent set of G' of size at least $(1 - \gamma') \cdot k$,
- i be a vertex of H, and suppose its degree is 3 (the case of vertices of smaller degree is treated analogously),
- j_1, j_2, j_3 be the neighbors of i in H,
- A_i be an intersection of I and vertices of cliques in $\mathcal{C}_i$.

We claim that if $|A_i| = 3s + 3$ (i.e., I intersects each clique in $\mathcal{C}_i$), then $r_i(v_1) = r_i(v_2) = r_i(v_3)$. Let $\widetilde{D}_p, \widetilde{D}_{p+1}$ be two consecutive cliques in the cycle $\mathcal{C}_i$ (it can be $\widetilde{D}_p = D^s_{ij_3}$ and $\widetilde{D}_{p+1} = C_{ij_1}$). Note that two cliques in $\mathcal{C}_i$ are adjacent if and only if they are consecutive.

Let $v'_p = I \cap V(\widetilde{D}_p)$. Define a relation $\succeq_i$ on V_i, such that $u \succeq_i v$ iff $u \not\prec_i v$. Since $\prec_i$ is a total order on V_i, we have that $u \succeq_i v$ iff $v \prec_i u$ or $u = v$. Since $v'_1, \ldots, v'_{3s+3}$ is an independent set, it holds that $r_i(v'_1) \succeq_i r_i(v'_2) \succeq_i \cdots \succeq_i r_i(v'_{3s+3}) \succeq_i r_i(v'_1)$ by construction. This implies that all vertices v'_p represent the same vertex $w \in V_i$.

Now, if $|A_i| = 3s + 3$, we define $\phi(i) = w$ (where w is as in the previous paragraph). If $|A_i| < 3s + 3$ we define $\phi(i)$ arbitrarily. Vertices $i' \in H$ of degree 2 are processed similarly, however the size of $A_{i'}$ is compared to value $2s + 2$.

[4] The possibilities for $\{D_1, D_2\}$ are: $\{C_{ij}, D^1_{i,j}\}$ or $\{D^p_{ij}, D^{p+1}_{ij}\}$ for $p < s$.

We say that the set A_i is complete if $|A_i| = (s+1) \cdot \deg(i)$. Thus, if A_i and A_j are complete, then $\phi(i)\phi(j)$ is an edge of G.

Let $B \subseteq V(H)$ be a set of vertices i of H such that A_i is not complete. Note that a primary clique C_{ij} is in two vertex-cycles of base cliques $\mathcal{C}_i$ and $\mathcal{C}_j$ and each secondary clique is in exactly one vertex-cycle of base cliques. Since there are fewer than $\gamma' \cdot k$ base cliques D such that $I \cap D = \emptyset$, the set B has size less than $2\gamma' \cdot k$. The vertices in B are incident to at most $6\gamma' \cdot k$ edges in H, and all remaining edges of H are satisfied by ϕ. Therefore,

$$\mathrm{val}(\Gamma) \geq \frac{|E(H)| - 6\gamma' \cdot k}{|E(H)|} = 1 - 6\gamma' \cdot \left(1 + \frac{s}{2}\right) = 1 - \gamma. \qquad \square$$

This completes the proof of Theorem 7 in this case.

5.2 $(C_5 \ldots, C_z, K_{1,4})$-free graphs

In this section we show Theorem 7 for $\mathcal{C}([5,z],4)$, i.e., for $(C_5 \ldots, C_z, K_{1,4})$-free graphs. The proof is similar to the case of $\mathcal{C}^*([4,z],5)$. Let $\Gamma = (G, V_1, \ldots, V_\ell, H)$ be an instance of MCSI(3), we will create an instance (G', k) of Independent Set, where $G' \in \mathcal{C}(5, z, 4)$. Consider an edge ij of H. We introduce four *primary cliques* $C^1_{ij}, C^2_{ij}, C^3_{ij}, C^4_{ij}$, each of size $|E_{ij}|$. For each $q \in [4]$, each vertex v of C^q_{ij} *represents* one edge in E_{ij}, denote this edge by $r'(v)$.

For each $q \in [4]$, we create $s := \lceil (z-3)/4 \rceil$ copies of C^q_{ij}, denoted by $D^{q,1}_{ij}, \ldots, D^{q,s}_{ij}$. Each vertex of a copy represents the same edge as the corresponding vertex in C^q_{ij}. The cliques created in this step will be called *cycle cliques*. Again, we imagine that the primary and cycle cliques are arranged in a cyclic way and constitute the *edge-cycle* corresponding to ij:

$$C^1_{ij}, D^{1,1}_{ij}, \ldots, D^{1,s}_{ij}, C^2_{ij}, D^{2,1}_{ij}, \ldots, D^{2,s}_{ij}, C^3_{ij}, D^{3,1}_{ij}, \ldots, D^{3,s}_{ij}, C^4_{ij}, D^{4,1}_{ij}, \ldots, D^{4,s}_{ij}, C^1_{ij}.$$

Note that all cliques in the edge-cycle are identical. We fix some arbitrary ordering $\prec_{ij}$ on E_{ij}, For each two consecutive cliques D_1 and D_2 of the edge-cycle, where D_1 precedes D_2, and for any vertex v from D_1 and any vertex w from D_2, we make vw adjacent in G' if and only if $r'(v) \prec_{ij} r'(w)$.

After repeating the previous step for every edge ij of H, we arrive at the point that G' consists of separate edge-cycles, one for each edge of H. Since H has maximum degree 3, each edge of H intersects at most 4 other edges. So for each pair of intersecting edges ia and ib we can assign a pair of primary cliques, one in the edge-cycle corresponding to ia, and the other one in the edge-cycle corresponding to ib, so that no primary clique is assigned twice.

Consider two edges of H, that share a vertex, say edges ia and ib, and suppose the primary cliques chosen in the last step are C^p_{ia} and C^q_{ib}. We need to provide some connection between these cliques, to make the choices for edges ia and ib consistent. Let us arbitrarily choose one of cliques C^p_{ia} and C^q_{ib}, say C^p_{ia}, and create s copies of it, denote these cliques by $F^1_{iab}, F^2_{iab}, \ldots, F^s_{iab}$ (again, the represented edges are inherited from the primary clique). We call these cliques *equality cliques*. We build an *equality gadget* by arranging these cliques in a sequence as follows:

$$C^p_{ia}, F^1_{iab}, F^2_{iab}, \ldots, F^s_{iab}, C^q_{ib}.$$

Consider two consecutive cliques D_1 and D_2 of this sequence, except for the last pair. These cliques are identical. Between them we add edges that form an antimatching, i.e., for a vertex v of D_1 and a vertex w of D_2, we add an edge vw if and only if $r'(v) \neq r'(w)$. Finally, for a vertex v of F^s_{iab} and a vertex w of C^q_{ib}, we add an edge vw if and only if $r'(v) \cap r'(w) \neq \emptyset$, i.e., edges represented by these vertices contain different vertices from V_i.

This completes the construction of G'. By *base cliques* we mean primary cliques, cycle cliques, and equality cliques. Let k be the number of all base cliques, i.e.,

$$k := \underbrace{4|E(H)|}_{\substack{\text{primary}\\ \text{cliques}}} + \underbrace{4s|E(H)|}_{\substack{\text{cycle}\\ \text{cliques}}} + \underbrace{\sum_{i \in V(H)} \binom{\deg_H(i)}{2} \cdot s}_{\text{equality cliques}} = \mathcal{O}(\ell).$$

Let us upper-bound k. If ℓ_2 and ℓ_3 are, respectively, the numbers of vertices of H with degree 2 and 3, then we obtain

$$k = 4|E(H)|(s+1) + s(\ell_2 + 3\ell_3) \leq \frac{9s}{2} \cdot |E(H)| + 4 \leq 5s \cdot |E(H)|. \tag{1}$$

The following claim is proven in an analogous way to Claim 5.2, note that this time we might obtain induced copies of C_4, where two vertices are in an equality clique, and the other two are in a different base clique in the same equality gadget (either an equality clique or a primary clique).

Claim 5.7. *The graph G' is $(C_5, \ldots, C_z)$-free.*

The next claim is in turn analogous to Claim 5.3.

Claim 5.8. *The graph G' is $K_{1,4}$-free.*

Proof. Observe that each clique is adjacent to at most three other cliques, and the only cliques adjacent to three other cliques are primary cliques. So if we hope to find an induced $K_{1,4}$, the center and one leaf must be in a primary clique, say C^q_{ij}, and other three leaves are in distinct base cliques adjacent to C^q_{ij}. However, two of cliques adjacent to C^q_{ij} must belong to the same edge-cycle (and the third one is an equality clique). Similarly as in the proof of Claim 5.3, we observe that the leaf that belongs to C^q_{ij} must be adjacent to at least one of the remaining leaves. □

The following claims are analogous to the corresponding claims in Section 5.1. Therefore we provide only sketches of proofs.

Claim 5.9. *If* $\mathrm{val}(\Gamma) = 1$*, then the graph G' has an independent set of size k.*

Proof. Consider a solution ϕ of $\mathit{\Gamma}$ of value 1. Therefore, for each $ij \in E(H)$, the pair $\phi(i)\phi(j)$ is an edge of G. Note that this edge is represented by some v in each primary clique C^q_{ij}. We select those vertices to the set I. Recall that each remaining clique C (i.e., a cycle clique or an equality clique), is a copy of some primary clique C'. For each such clique C we include to I the vertex, which is a copy of the selected vertex in C'.

By an argument analogous to the one in the proof of Claim 5.5 we observe that the selected vertices belonging to one edge-cycle are pairwise non-adjacent. Furthermore, note that the edges between adjacent cliques in an equality gadget are defined in a way, so that all selected vertices from cliques in this gadget are pairwise non-adjacent. Thus, the I is an independent set of size k. □

Claim 5.10. *Let $\gamma > 0$. If the graph G' has an independent set of size at least $(1-\gamma') \cdot k$ for $\gamma' = \frac{\gamma}{45s}$, then val$(\mathit{\Gamma}) \geq 1 - \gamma$.*

Proof. Consider an independent set I in G of size at least $(1-\gamma') \cdot k$, and a vertex $i \in V(H)$. Suppose that $\deg(i) = 3$ and the neighbors of i in H are a, b, c (if the degree of i is smaller, the reasoning is analogous).

Let $\mathcal{S}^i$ be the union of all base cliques corresponding to i, i.e.,

1. belonging to edge-cycles corresponding to ia, ib, ic, and
2. belonging to equality gadgets between these edge-cycles.

Note that the number of cliques in $\mathcal{S}^i$ is $3 \cdot 4(s+1) + 3 \cdot s = 15s + 12$, and let A^i be the intersection of I with the vertices of $\mathcal{S}^i$. Suppose that the size of A^i is $15s + 12$, i.e., we selected a vertex from each base clique in $\mathcal{S}$ – we call such A^i complete. By the reasoning analogous to Claim 5.6, we observe that for each of three edge-cycles in $\mathcal{S}^i$, the selected vertices correspond to the same edge of G, denote these edges by e_1, e_2, e_3, respectively. Furthermore, as in the proof of Claim 5.9, we observe that the edges e_1, e_2, e_3 share a vertex $v \in V_i$. If A^i is complete, we set $\phi(i) = v$. Otherwise, we set $\phi(i)$ arbitrarily.

Let B be the set of those i, for which A^i is not complete. We observe that each base clique C is in at most three sets $\mathcal{S}^i$. Consider a base clique C. If C is a primary clique or a cycle clique, then it corresponds to some E_{ij}, and C belongs $\mathcal{S}^i$ and $\mathcal{S}^j$. In the last case, if C is an equality clique in the equality gadget joining edge-cycles corresponding to ia and ib, then C belongs to $\mathcal{S}^i, \mathcal{S}^a, \mathcal{S}^b$. Summing up, each base clique belongs to at most three sets $\mathcal{S}^i$. Since there are fewer than $\gamma' \cdot k$ base clique C, such that $C \cap I = \emptyset$, we observe that the size of B is at most $3\gamma' \cdot k$. The vertices in B are incident to at most $9\gamma' \cdot k$ edges in H, and all remaining edges are satisfied by ϕ. So, using (1), we obtain

$$\text{val}(\mathit{\Gamma}) \geq \frac{|E(H)| - 9\gamma' \cdot k}{|E(H)|} \geq 1 - 45s \cdot \gamma' = 1 - \gamma. \qquad \square$$

6 Parameterized approximation with H as a parameter

In this section we still consider the Independent Set problem in H-free graphs, but now our parameter is related to the graph H. First, we show Proposition 1. We point out that a similar argument was also observed by Bonnet [3].

Proposition 1. *Let d be an integer and let $\mathcal{H}_d$ be a family of graphs, such that $\alpha(H) > d$ for every $H \in \mathcal{H}_d$. The* Independent Set *problem in $\mathcal{H}_d$-free graphs is* W[1]*-hard parameterized by d and cannot be solved in $n^{o(d)}$ time, unless the ETH fails.*

Proof. Let $\mathcal{H}_d$ be a family of graphs as in the statement. We will reduce from k-Multicolored Independent Set, which is W[1]-hard and has no $n^{o(k)}$ algorithm, unless the ETH fails [12, Theorem 13.25 and Corollary 14.23]. Set $k = d$ and let G be an instance of k-Multicolored Independent Set, with $V(G)$ partitioned into k pairwise disjoint sets $V_1, V_2 \ldots, V_k$. We define G' to be the graph obtained from G by turning each set V_i into a clique. It is straightforward to see that G is a yes-instance of k-Multicolored Independent Set if and only if G' has an independent set of size k. Furthermore, let us observe that the vertex set of G' is partitioned into $k = d$ cliques, so G' is clearly H-free for every $H \in \mathcal{H}_d$. □

Now let us consider the Independent Set problem in $K_{1,d}$-free graphs, parameterized by *both* k and d. In this case we are able to give parameterized approximation lower bounds based on the following sparsification of MCSI. Recall that $\xi(\ell) = 2^{(\log \ell)^{1/2+\varepsilon}} = \ell^{o(1)}$ for any constant $\frac{1}{2} > \varepsilon > 0$, i.e., the term grows slower than any polynomial (but faster than any polylogarithm).

Theorem 9. *Consider an instance $\Gamma = (G, V_1, \ldots, V_\ell, J)$ of* MCSI(t) *with size n and $t > \xi(\ell)$. Assuming the deterministic Gap-ETH, for any computable function f, there is no $f(\ell) \cdot n^{O(1)}$ time algorithm that can distinguish between the two cases:*
1. *(YES-case)* $\text{val}(\Gamma) = 1$*, and*
2. *(NO-case)* $\text{val}(\Gamma) \leq \xi(\ell)/t$*.*

To prove Theorem 9 we need two facts. The first is the Erdős-Gallai theorem on *degree sequences*, which are sequences of non-negative integers $d_1, \ldots, d_n$, for each of which there exists a simple graph on n vertices such that vertex $i \in [n]$ has degree d_i. We use the following constructive formulation due to Choudum [9].

Theorem 10 (Erdős-Gallai theorem [9]). *A sequence of non-negative integers $d_1 \geq \cdots \geq d_n$ is a degree sequence of a simple graph on n vertices if $d_1 + \cdots + d_n$ is even and for every $1 \leq k \leq n$ the following inequality holds: $\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min(d_i, k)$. Moreover, given such a degree sequence, a corresponding graph can be constructed in polynomial time.*

We also need a parameterized approximation lower bound for MCSI, as given by Dinur and Manurangsi [14].

Theorem 11 (Dinur and Manurangsi [14]). *Consider an instance $\Gamma = (G, V_1, \ldots, V_\ell, J)$ of* MCSI *with size n and J a complete graph. Assuming the deterministic Gap-ETH, there is no $f(\ell) \cdot n^{O(1)}$ time algorithm for any computable function f, that can distinguish between the following two cases:*
1. *(YES-case)* $\text{val}(\Gamma) = 1$*, and*

2. *(NO-case)* $\mathrm{val}(\Gamma) \leq \xi(\ell)/\ell$.

Proof of Theorem 9. Let $t < \ell$ and let $\Gamma = (G, V_1, \ldots, V_\ell, J)$ be an instance of MCSI where J is a complete graph. To find an instance of MCSI(t), we want to construct a graph J' on ℓ vertices with maximum degree t, for which we use the Erdős-Gallai theorem. By Theorem 10 it is easy to verify that a t-regular graph on ℓ vertices exists if $t\ell$ is even. However, if $t\ell$ is odd, there is a graph with $\ell - 1$ vertices of degree t and one vertex of degree $t - 1$. Moreover, the proof of Theorem 10 by Choudum [9] is constructive, and gives a polynomial time algorithm. Hence we can compute a graph J' with maximum degree t and $|E(J')| \geq (t\ell - 1)/2$. Since J is a complete graph, J' is a subgraph of J.

We create a graph G' by removing edges from G according to J': we remove all edges between sets V_i and V_j of G if and only if $ij \notin E(J')$, and call the resulting graph G'. Thus, we get new instance $\Gamma' = (G', V_1, \ldots, V_\ell, J')$ of MCSI(t).

It is easy to see that if $\mathrm{val}(\Gamma) = 1$, then $\mathrm{val}(\Gamma') = 1$ as well: we just use the optimal solution for Γ. Now suppose that $\mathrm{val}(\Gamma) \leq \nu$, which means that each solution ϕ satisfies at most a ν-fraction of edges of J. Let ϕ be an arbitrary solution of Γ', which is also a solution for Γ because $V(G) = V(G')$ and $V(J) = V(J')$. By our assumption we know that it satisfies at most $\nu \cdot E(J)$ edges of J. Thus, the solution ϕ satisfies at most $\nu \cdot E(J)$ edges of J' as well. Hence we obtain

$$\mathrm{val}(\Gamma') \leq \frac{\nu \cdot E(J)}{E(J')} = \nu \cdot \frac{\ell(\ell-1)}{t\ell - 1} \leq \nu \cdot \frac{\ell}{t - 1/\ell}.$$

Now, by Theorem 11 we know that under the deterministic Gap-ETH, no $f(\ell) \cdot n^{O(1)}$ time algorithm can distinguish between $\mathrm{val}(\Gamma) = 1$ and $\mathrm{val}(\Gamma) \leq \xi(\ell)/\ell$ given Γ. By the above calculations, for Γ' we obtain that no such algorithm can distinguish between $\mathrm{val}(\Gamma') = 1$ and $\mathrm{val}(\Gamma') \leq \frac{\xi(\ell)}{t-1/\ell}$ by setting $\nu = \xi(\ell)/\ell$. Recall that $\xi(\ell) = 2^{(\log \ell)^{1/2+\varepsilon}}$ where ε can be set to any positive constant in Theorem 11. Given any constant $\varepsilon' > 0$, we choose ε such that $2^{(\log \ell)^{1/2+\varepsilon}}/2^{(\log \ell)^{1/2+\varepsilon'}} \leq (t-1/\ell)/t$. It can be verified that such a constant $\varepsilon > 0$ always exists, assuming w.l.o.g. that ℓ is larger than some sufficiently large constant. This implies that $\mathrm{val}(\Gamma') \leq \frac{\xi(\ell)}{t-1/\ell} \leq 2^{(\log \ell)^{1/2+\varepsilon'}}/t$. Note that $\mathrm{val}(\Gamma') < 1$ if $t > 2^{(\log \ell)^{1/2+\varepsilon'}}$, and so we obtain Theorem 9 (for $\xi(\ell) := 2^{(\log \ell)^{1/2+\varepsilon'}}$). □

Based on Theorem 9 we can prove Proposition 2, using the same reduction as in Lemma 1.

Proposition 2. *Let $\varepsilon > 0$ be any constant and $\xi(k) = 2^{(\log k)^{1/2+\varepsilon}}$. The* Independent Set *problem in $K_{1,d}$-free graphs has no $d/\xi(k)$-approximation algorithm with runtime $f(d,k) \cdot n^{\mathcal{O}(1)}$ for any computable function f, unless the deterministic Gap-ETH fails.*

Proof. We reduce via Lemma 1 from MCSI(t) to Independent Set, which given an instance Γ of MCSI(t) results in a $K_{1,2t+2}$-free graph G for Independent Set. We thus set $d = 2t + 2$. If $\mathrm{val}(\Gamma) = 1$, then G has an independent set of size $k = \binom{\ell}{2}$. If $\text{val}(\Gamma) \leq \xi(\ell)/t$, then every independent set of G has size at most $\xi(\ell)\binom{\ell}{2}/t \leq \frac{\xi(k)k}{d/2-1}$, assuming w.l.o.g. that $k \geq 4$ so that $\ell \leq 2\sqrt{k} \leq k$. Given a constant $\varepsilon' > 0$ we may choose ε small enough in Theorem 9 so that $\frac{\xi(k)k}{d/2-1} \leq 2^{(\log k)^{1/2+\varepsilon'}} k/d$. Thus, for $\xi'(k) = 2^{(\log k)^{1/2+\varepsilon'}}$, a $d/\xi'(k)$-approximation algorithm for Independent Set would be able to distinguish between the two cases of Γ. Note that $d = 2t + 2 \leq 2\ell$ as the maximum degree of the graph J is $\ell - 1$. Thus if the runtime of this algorithm is $f(d, k) \cdot n^{\mathcal{O}(1)}$, then for some function f' this would be a $f'(\ell) \cdot n^{\mathcal{O}(1)}$ time algorithm for MCSI(t). However, according to Theorem 9 this would be a contradiction, unless the deterministic Gap-ETH fails. We may rename $\xi'(k)$ to $\xi(k)$ to obtain Proposition 2. □

7 Conclusion and open problems

Our parameterized inapproximability results of Theorem 5 suggest that the Independent Set problem is hard to approximate to within some constant, whenever it is W[1]-hard to solve on H-free graphs, according to Theorem 2. In most cases it is unclear though whether any approximation can be computed (either in polynomial time or by exploiting the parameter k), which beats the strong lower bounds for polynomial-time algorithms for general graphs. The only known exceptions to this are the $K_{1,d}$-free case, where a polynomial-time $(\frac{d-1}{2} + \delta)$-approximation algorithm was shown by Halldórsson [20], and the $K_{a,b}$-free case, for which we showed a polynomial-time $\mathcal{O}\big((a+b)^{1/a} \cdot \alpha(G)^{1-1/a}\big)$-approximation algorithm in Theorem 3. For $K_{1,d}$-free graphs, we were also able to show an almost asymptotically tight lower bound for polynomial-time algorithms in Theorem 4. For parameterized algorithms, our lower bound of Proposition 2 for $K_{1,d}$-free graphs does not give a tight bound, but seems to suggest that parameterizing by k does not help to obtain an improvement. For P_t-free graphs, for which the Independent Set problem is conjectured to be polynomial-time solvable, we showed in Proposition 1 that the complexity of any such algorithm must grow with the length t of the excluded path.

Settling the question whether H-free graphs admit better approximations to Independent Set than general graphs, remains a challenging open problem, both for polynomial-time algorithms and algorithms exploiting the parameter k.

Let us point out one more, concrete open question. Recall from Theorem 2 Bonnet *et al.* [4] were able to show W[1]-hardness for graphs which *simultanously* exclude $K_{1,4}$ and all induced cycles of length in $[4, z]$, for any constant $z \geq 5$. On the other hand, we presented two separate reductions, one for $(K_{1,5}, C_4, \ldots, C_z)$-free graphs, and another one for $(K_{1,4}, C_5, \ldots, C_z)$-free graphs. It would be nice to provide a uniform reduction, i.e., prove hardness for parameterized approximation in $(K_{1,4}, C_4, \ldots, C_z)$-free graphs.

References

1. V. Alekseev. Polynomial algorithm for finding the largest independent sets in graphs without forks. *Discrete Applied Mathematics*, 135(1):3 – 16, 2004. Russian

Translations II.
2. V. E. Alekseev. The effect of local constraints on the complexity of determination of the graph independence number. *Combinatorial-algebraic methods in applied mathematics*, pages 3–13, 1982.
3. É. Bonnet. private communication.
4. É. Bonnet, N. Bousquet, P. Charbit, S. Thomassé, and R. Watrigant. Parameterized complexity of independent set in H-free graphs. In C. Paul and M. Pilipczuk, editors, *13th International Symposium on Parameterized and Exact Computation, IPEC 2018, August 20-24, 2018, Helsinki, Finland*, volume 115 of *LIPIcs*, pages 17:1–17:13. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2018.
5. É. Bonnet, N. Bousquet, S. Thomassé, and R. Watrigant. When maximum stable set can be solved in FPT time. In P. Lu and G. Zhang, editors, *30th International Symposium on Algorithms and Computation, ISAAC 2019, December 8-11, 2019, Shanghai University of Finance and Economics, Shanghai, China*, volume 149 of *LIPIcs*, pages 49:1–49:22. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2019.
6. P. Chalermsook, M. Cygan, G. Kortsarz, B. Laekhanukit, P. Manurangsi, D. Nanongkai, and L. Trevisan. From Gap-ETH to FPT-Inapproximability: Clique, Dominating Set, and More. In *58th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2017, Berkeley, CA, USA, October 15-17, 2017*, pages 743–754, 2017.
7. S. Chawla, editor. *Proceedings of the 2020 ACM-SIAM Symposium on Discrete Algorithms, SODA 2020, Salt Lake City, UT, USA, January 5-8, 2020*. SIAM, 2020.
8. R. Chitnis, A. E. Feldmann, and P. Manurangsi. Parameterized approximation algorithms for bidirected Steiner Network problems, 2017.
9. S. Choudum. A simple proof of the Erdős-Gallai theorem on graph sequences. *Bulletin of the Australian Mathematical Society*, 33(1):67âĂŞ70, 1986.
10. M. Chudnovsky, M. Pilipczuk, M. Pilipczuk, and S. Thomassé. Quasi-polynomial time approximation schemes for the maximum weight independent set problem in H-free graphs. In Chawla [7], pages 2260–2278.
11. D. Corneil, H. Lerchs, and L. Burlingham. Complement reducible graphs. *Discrete Applied Mathematics*, 3(3):163 – 174, 1981.
12. M. Cygan, F. V. Fomin, L. Kowalik, D. Lokshtanov, D. Marx, M. Pilipczuk, M. Pilipczuk, and S. Saurabh. *Parameterized Algorithms*. Springer, 2015.
13. K. Dabrowski, V. V. Lozin, H. Müller, and D. Rautenbach. Parameterized algorithms for the independent set problem in some hereditary graph classes. In C. S. Iliopoulos and W. F. Smyth, editors, *Combinatorial Algorithms - 21st International Workshop, IWOCA 2010, London, UK, July 26-28, 2010, Revised Selected Papers*, volume 6460 of *Lecture Notes in Computer Science*, pages 1–9. Springer, 2010.
14. I. Dinur and P. Manurangsi. ETH-hardness of approximating 2-CSPs and Directed Steiner Network. In A. R. Karlin, editor, *9th Innovations in Theoretical Computer Science Conference, ITCS 2018, January 11-14, 2018, Cambridge, MA, USA*, volume 94 of *LIPIcs*, pages 36:1–36:20. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2018.
15. I. Dinur and P. Manurangsi. ETH-hardness of approximating 2-CSPs and Directed Steiner Network. *CoRR*, abs/1805.03867, 2018.
16. P. Erdős and G. Szekeres. *A Combinatorial Problem in Geometry*, pages 49–56. Birkhäuser Boston, Boston, MA, 1987.
17. U. Feige. Approximating maximum clique by removing subgraphs. *SIAM J. Discrete Math.*, 18(2):219–225, 2004.

18. M. Garey, D. Johnson, and L. Stockmeyer. Some simplified NP-complete graph problems. *Theoretical Computer Science*, 1(3):237 – 267, 1976.
19. A. Grzesik, T. Klimosova, M. Pilipczuk, and M. Pilipczuk. Polynomial-time algorithm for maximum weight independent set on p6-free graphs. In T. M. Chan, editor, *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2019, San Diego, California, USA, January 6-9, 2019*, pages 1257–1271. SIAM, 2019.
20. M. M. Halldórsson. Approximating discrete collections via local improvements. In K. L. Clarkson, editor, *Proceedings of the Sixth Annual ACM-SIAM Symposium on Discrete Algorithms, 22-24 January 1995. San Francisco, California, USA*, pages 160–169. ACM/SIAM, 1995.
21. J. Håstad. Clique is hard to approximate within $n^{(1-\varepsilon)}$. In *Acta Mathematica*, pages 627–636, 1996.
22. R. M. Karp. Reducibility among combinatorial problems. In *Complexity of Computer Computations*, pages 85–103. Springer US, 1972.
23. S. Khot and A. K. Ponnuswami. Better inapproximability results for Max Clique, Chromatic Number and Min-3Lin-Deletion. In M. Bugliesi, B. Preneel, V. Sassone, and I. Wegener, editors, *Automata, Languages and Programming*, pages 226–237, Berlin, Heidelberg, 2006. Springer Berlin Heidelberg.
24. B. Laekhanukit. Parameters of two-prover-one-round game and the hardness of connectivity problems. In *Proceedings of the twenty-fifth annual ACM-SIAM symposium on Discrete algorithms (SODA)*, pages 1626–1643. SIAM, 2014.
25. D. Lokshantov, M. Vatshelle, and Y. Villanger. Independent set in P_5-free graphs in polynomial time. In C. Chekuri, editor, *Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2014, Portland, Oregon, USA, January 5-7, 2014*, pages 570–581. SIAM, 2014.
26. D. Lokshtanov, M. S. Ramanujan, S. Saurabh, and M. Zehavi. Parameterized complexity and approximability of directed odd cycle transversal. *CoRR*, abs/1704.04249, 2017.
27. V. V. Lozin and M. Milanic. A polynomial algorithm to find an independent set of maximum weight in a fork-free graph. *J. Discrete Algorithms*, 6(4):595–604, 2008.
28. P. Manurangsi. Tight running time lower bounds for strong inapproximability of maximum k-coverage, unique set cover and related problems (via t-wise agreement testing theorem). In Chawla [7], pages 62–81.
29. D. Marx. Can you beat treewidth? *Theory of Computing*, 6(1):85–112, 2010.
30. D. Marx and M. Pilipczuk. Optimal parameterized algorithms for planar facility location problems using voronoi diagrams. *CoRR*, abs/1504.05476, 2015.
31. G. J. Minty. On maximal independent sets of vertices in claw-free graphs. *Journal of Combinatorial Theory, Series B*, 28(3):284 – 304, 1980.
32. N. Sbihi. Algorithme de recherche d'un stable de cardinalite maximum dans un graphe sans etoile. *Discrete Mathematics*, 29(1):53 – 76, 1980.