Discrete Mathematics

Exercise sheet 8

21/ 27 November 2016

1. Last week we read the sorry tale of the restaurant cook who lost both her engagement ring and her wedding ring in a large pot of soup, all of which was served up evenly among 25 guests. In a development of the plot, upon receiving his plate of soup one of the guests huffily declared it to be a "dog's dinner" and thereupon gave it to the restaurant dog sitting hungrily in the corner. What is the probability that

(a) the dog finds the wedding ring in its soup, given that it also finds the engagement ring?

Let A_1 be the event that the dog finds the engagement ring in its soup, and A_2 the event that it finds the wedding ring in its soup. Then $A_1 \cap A_2$ is the event that the dog has both rings in its soup, and $A_1 \cup A_2$ the event that it finds at least one ring in its soup. We have

$$\mathbb{P}(A_1) = \frac{1}{25} = \mathbb{P}(A_2).$$

By independence of the events A_1 and A_2,

$$\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2) = \frac{1}{25} \cdot \frac{1}{25} = \frac{1}{625}.$$

Thus

$$\mathbb{P}(A_2|A_1) = \frac{\mathbb{P}(A_1 \cap A_2)}{\mathbb{P}(A_2)} = \frac{\frac{1}{625}}{\frac{25}{625}} = \frac{1}{25}.$$

(b) the dog finds both rings in its soup, given that it finds at least one?

We have

$$\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2) = \frac{1}{25} + \frac{1}{25} - \frac{1}{625} = \frac{49}{625}.$$

Thus,

$$\mathbb{P}(A_1 \cap A_2 \,|\, A_1 \cup A_2) = \frac{\mathbb{P}(A_1 \cap A_2)}{\mathbb{P}(A_1 \cup A_2)} = \frac{\frac{1}{625}}{\frac{49}{625}} = \frac{1}{49}.$$

2. In a random string of one hundred bits, in which each bit takes value 0 or 1 with equal probability, and different bits are independent:

(a) Determine the probability that a fixed set of 6 consecutive bits of the string are all equal to 1.

Let A_i be the event that six consecutive bits in positions $i, i+1, i+2, i+3, i+4, i+5$ are equal to 1. (Here i is between 1 and 95, A_1 being the event that first six bits are 1, A_95 being that the last six bits are 1.)

Since bits are independent,

$$\mathbb{P}(A_i) = \left(\frac{1}{2}\right)^6 = \frac{1}{64}$$

for any fixed i.

(b) Find the expected number of 6 consecutive 1s in the string (i.e., number of substrings 111111). [*Hint: introduce indicator variables for an occurrence of substring* 111111 *beginning at position* i $(1 \le i \le 95)$ *and use linearity of expectation.*]

Let X_i be the random variable equal to 1 when A_i occurs, and equal to 0 otherwise (the *indicator* variable of the event A_i). Then

$$\mathbb{E}(X_i) = 1 \cdot \frac{1}{64} + 0 \cdot \frac{63}{64} = \frac{1}{64}.$$

and the expected number of appearances of 111111 (six consecutive 1s) is

$$\mathbb{E}\left(\sum_{i=1}^{95} X_i\right) = \sum_{i=1}^{95} \mathbb{E}(X_i) = 95 \cdot \frac{1}{64} = \frac{95}{64},$$

where we have used linearity of expectation in the first equality.

(c) More generally, for $0 \le k \le n$, in a random string of n independent bits, the values 0 and 1 occurring with equal probability in each position, determine the expected number of k consecutive 1s.

Define A_i for $i = 1, \ldots, n-k+1$ to be the event that the bits in positions $i, i+1, \ldots, i+k-1$ are all equal to 1 and X_i the indicator variable of A_i. Then $\mathbb{P}(A_i) = 2^{-k}$ and $\mathbb{E}(X_i) = 2^{-k}$. Hence the expected number of occurrences of A_i among $i = 1, 2, \ldots, n-k+1$ is equal to

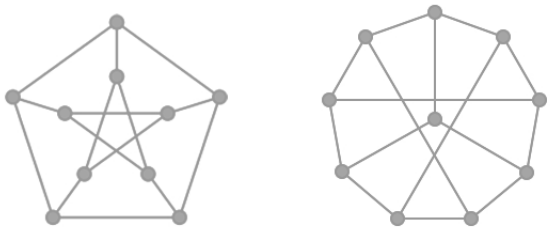
$$\mathbb{E}\left(\sum_{i=1}^{n-k+1} X_i\right) = \sum_{i=1}^{n-k+1} \mathbb{E}(X_i) = (n-k+1) \cdot 2^{-k} = \frac{n-k+1}{2^k}.$$

(d) A *switch* occurs when consecutive bits are different. Determine the expected number of switches in a random string of n bits.

Define B_i for $i = 1, 2, \ldots, n-1$ to be the event that there is a switch of value from position i to position $i+1$. Then $\mathbb{P}(B_i) = \frac{2}{4}$ (two outcomes 10 and 01 out of four possible 2-bit strings). Letting X_i be the indicator of B_i, the expected number of switches is then

$$\mathbb{E}\left(\sum_{i=1}^{n-1} X_i\right) = \sum_{i=1}^{n-1} \mathbb{E}(X_i) = (n-1) \cdot \frac{1}{2} = \frac{n-1}{2}.$$

3. Find an isomorphism between the following graphs:



What length cycles do these graphs have?
Cycles of lengths 5,6,8,9.
[The graph pictured is called the *Petersen graph.*]

4. Explain why no two of the following graphs are isomorphic:
The first graph has no 4-cycles, the second has 4-cycles but not sharing any edge, the third has neighbouring 4-cycles.
What property do the three graphs have in common (other than each having 10 vertices)?
each vertex degree 3, same number of edges, connected, ...