

# On Packet Scheduling with Adversarial Jamming and Speedup\*

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## Abstract

In Packet Scheduling with Adversarial Jamming packets of arbitrary sizes arrive over time to be transmitted over a channel in which instantaneous jamming errors occur at times chosen by the adversary and not known to the algorithm. The transmission taking place at the time of jamming is corrupt, and the algorithm learns this fact immediately. An online algorithm maximizes the total size of packets it successfully transmits and the goal is to develop an algorithm with the lowest possible asymptotic competitive ratio, where the additive constant may depend on packet sizes.

Our main contribution is a simple algorithm that works for any speedup and packet sizes and, unlike previous algorithms for the problem, it does not need to know these properties in advance. We show that this algorithm maintains 1-competitiveness with speedup 4, making it the first known algorithm to maintain 1-competitiveness with a moderate speedup in the general setting of arbitrary packet sizes. Furthermore, we show a lower bound of  $\phi + 1 \approx 2.618$  on the speedup of any 1-competitive deterministic algorithm, showing that our algorithm is close to the optimum.

Additionally, we formulate a general framework for analyzing our algorithm locally and use it to show upper bounds on its competitive ratio for speedups in  $[1, 4)$  and for several special cases. In particular, our algorithm is 3-competitive without speedup, matching the algorithm and the lower bound of Jurdzinski *et al.* [7]. We use this framework also for the case of divisible packet sizes in which the size of a packet divides the size of any larger packet, to show that a slight modification of our algorithm is 1-competitive with speedup 2 and it achieves the optimal competitive ratio of 2 without speedup, again matching the algorithm and the lower bound of [7]. Finally, we obtain a lower bound of 2 on the speedup of any 1-competitive deterministic algorithm using only two divisible packet sizes, which is tight.

## 1 Introduction

We study an online packet scheduling model recently introduced by Anta *et al.* [1] and extended by Jurdzinski *et al.* [7]. In our model, packets of arbitrary sizes arrive over time. The algorithm schedules any packet of its choice at any time, but cannot interrupt its subsequent transmission; in the scheduling jargon, there is a single machine and no preemptions. There are, however, instantaneous *jamming errors* or *faults* at times chosen by the adversary, which are not known to the algorithm. A transmission taking place at the time of jamming is corrupt, and the algorithm learns this fact immediately. The packet whose transmission failed can be retransmitted immediately or at any later time.

The objective is to maximize the total size of packets successfully transmitted. In particular, the goal is to develop an online algorithm with the lowest possible competitive ratio. We focus on algorithms with resource augmentation, namely on online algorithms that transmit packets  $s \geq 1$  times faster than the offline optimum solution they are compared against; such algorithm is often said to be speed- $s$ , running at speed  $s$ , or having a speedup of  $s$ . We note that this notion has been introduced for a similar job scheduling problem [8]. However, our problem allows constant competitive ratio already without any speedup, and hence we focus on determining the speed that is necessary

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and sufficient for 1-competitiveness, i.e., matching the optimum speed-1 performance. In contrast to previous work that followed this avenue, we also study the tradeoff between the competitive ratio and the speedup, i.e., the ratios that can be attained at intermediate speeds.

The distinguishing feature of the model is that the number of different packet sizes is a constant and that these packet sizes are considered to be constants. Thus, the additive term in the definition of the competitive ratio may depend on the number and values of the packet sizes. (It is easy to observe that, at speed 1, without such additive term, it is impossible to attain constant competitive ratio that does not depend on packet sizes, even for only two different sizes.)

## 1.1 Previous Results

The model was introduced by Anta *et al.* [1], who resolved it for two packet sizes: If  $\gamma > 1$  denotes the ratio of the two sizes, then the optimum ratio for deterministic algorithms is  $\frac{\gamma+1\lceil\gamma\rceil}{\lfloor\gamma\rfloor}$ , which is always in the range  $[2, 3)$ . Anta *et al.* [1] note that their lower bound strategy applies to randomized algorithms as well, but their argument only applies to the adaptive adversary model. To our best knowledge, randomized algorithms for the more common oblivious adversary model were never considered in this setting. We note that Anta *et al.* [1] demonstrate the necessity of instantaneous error feedback by proving that discovering errors upon completed transmission rules out constant competitive ratio. They also provide improved results for a stochastic online setting.

These results were extended by Jurdzinski *et al.* [7], who proved that the optimum ratio for the case of multiple (though fixed) packet sizes is given by the same formula for the two packet sizes which maximize it. Moreover, Jurdzinski *et al.* [7] gave further results for *divisible* packet sizes, i.e., instances in which every packet size divides every larger packet size, for which the optimum ratio is 2. Namely, they generalized the 2-competitive algorithm to a setting with  $f$  independent transmission channels (or machines in scheduling parlance); jamming errors on each channel are independent, and a packet can only be transmitted on one channel at a time. Furthermore, for the single channel divisible case, they proved that speed 2 is sufficient for 1-competitiveness in the resource augmentation setting.

## 1.2 Our results

Our main new results concern the analysis of the non-divisible case with speedup, where we show that speed 4 is sufficient for a competitive ratio of 1 (Section 4). However, our major contribution is a uniform algorithm that works well in every setting, together with a uniform analysis framework (Section 3). This contrasts with the results of Jurdzinski *et al.* [7], where each upper bound was attained by a dedicated algorithm with independently crafted analysis. Moreover, unlike those algorithms, which all require prior knowledge of all possible packet sizes and speedup, in the sense that there are different algorithms for speeds 1 and 2 (only divisible case), with none for intermediate speeds, our algorithm is oblivious in that it requires no such information. Furthermore, our algorithm is more appealing as it is significantly simpler and “work-conserving” or “busy”, i.e., transmitting some packet whenever there is one pending, which is desirable in practice. See Section 2 for the description of our algorithm.

To recover the 1-competitiveness for divisible instances at speed 2, we have to modify our algorithm slightly, as otherwise we can guarantee this ratio only at speed 2.5. This is to be expected, as divisible instances are a very special case. On the other hand, we prove that our original algorithm is 1-competitive on far broader class of “well-separated” instances at sufficient speed: If the ratio between two successive packet sizes (in their sorted list) is no smaller than  $\alpha \geq 1$ , our algorithm is 1-competitive if its speed is at least  $S_\alpha$ , which is a non-increasing function of  $\alpha$  such that  $S_1 = 4$  and  $\lim_{\alpha \rightarrow \infty} S_\alpha = 2$ ; see Section 3.2 for the precise definition of  $S_\alpha$ .

In Section 5 we complement these results with two lower bounds on speed that is necessary to achieve 1-competitiveness, which are the first such results for the problem. The first one proves that even for two divisible packet sizes, speed 2 is required to attain 1-competitiveness, establishing optimality of our modified algorithm and that of Jurdzinski *et al.* [7] for the divisible case. The second one strengthens this result by showing that for non-divisible instances with more packet sizes, speed  $\phi + 1 \approx 2.618$  is required for 1-competitiveness. Both results hold even if all packets are released together.

Finally, in Section 6 we demonstrate that our bounds on speeds sufficient for 1-competitiveness of our algorithm are tight, i.e., that (a) on general instances, it is no better than  $4/s$ -competitive for  $s \in [2, 4)$ , (b) on divisible instances, it is no better than  $4/3$ -competitive for  $s < 2.5$ , and (c) it is at least 2-competitive for  $s < 2$ , even for two divisible packet sizes.

### 1.3 Related Results

We first mention a few results concerning the model with adversarial jamming. Anta *et al.* [3] consider essentially the same problem but with a different objective: minimizing the number and/or the total size of pending (i.e., not yet transmitted) packets. They focus on multiple channels (or machines, as therein the problem is formulated as job scheduling), and investigate what speedup is necessary and sufficient for competitiveness. The work of Georgio *et al.* [6] considers the same problem without any speedup, restricted to jobs of unit size, under various efficiency and correctness measures, again focusing on multiple machines/channels. This article is motivated by distributed computation, and thus distinguishes between different information models and ways in which the machines communicate. In yet another work, Anta *et al.* [2] consider latency on top of the previous two objectives, and study the competitiveness of popular scheduling algorithms for all these measures. The problem on parallel channels was considered by Garncarek *et al.* [5] for two divisible packet sizes and “synchronized” channels that all suffer errors at the same time. This work distinguishes between “regular” jamming errors and “crashes”, which also cause the algorithm’s state to reset, losing any information stored about the past events and proves that the optimum ratio tends to  $\frac{4}{3}$  for the former and to  $\phi = (\sqrt{5} + 1)/2 \approx 1.618$  for the latter as the number  $f$  of channels tends to infinity.

As mentioned before, Kalyanasundaram and Pruhs [8] introduced resource augmentation. Among other results they proved that a constant competitive ratio is possible with a constant speedup for a preemptive variant of real-time scheduling where each job has a release time, deadline, processing time and a weight and the objective is to maximize the weight of jobs completed by their deadlines on a single machine. Subsequently resource augmentation was applied in various scenarios. Of the most relevant for us are those that considered algorithms with speedup that are 1-competitive, i.e., as good as the optimum. We mention two models that still contain interesting open problems.

For real-time scheduling, Phillips *et al.* [12] considered the underloaded case in which there exists a schedule that completes all the jobs. It is easy to see that on a single machine, the earliest deadline first (EDF) algorithm is then an optimal online algorithm. Phillips *et al.* [12] proved that EDF on  $m$  machines is 1-competitive with speedup  $2 - 1/m$ . (Here the weights are not relevant.) Intriguingly, finding a 1-competitive algorithm with minimal speedup is wide open: It is known that speedup at least  $6/5$  is necessary, it has been conjectured that speedup  $e/(e - 1)$  is sufficient, but the best upper bound proven is  $2 - 2/(m + 1)$  from [11]. See Schewior [13] for more on this problem.

Later these results were extended to real-time scheduling of overloaded systems, where for uniform density (i.e., weight equal to processing time) Lam *et al.* [10] have shown that a variant of EDF with admission control is 1-competitive with speedup 2 on a single machine and with speedup 3 on more machines. For non-uniform densities, the necessary speedup is a constant if each job is tight (its deadline equals its release time plus its processing time) [9]. Without this restriction it is no longer constant, depending on the ratio  $\xi$  of the maximum and minimum weight. It is known that it is at least  $\Omega(\log \log \xi)$  and at most  $O(\log \xi)$  [4, 10]; as far as we are aware, closing this gap is still an open problem.

## 2 Algorithms, preliminaries, notations

The general idea of the algorithm is that after each error, we start by transmitting packets of small sizes, only increasing the size of packets after a sufficiently long period of uninterrupted transmissions. It turns out that the right tradeoff is to transmit a packet only if it would have been transmitted successfully if started just after the last error. It is also crucial that the initial packet after each error has the right size, namely to ignore small packet sizes if the total size of not transmitted packets of those sizes is small compared to a larger packet that can be transmitted. This guarantees that if no error occurs, all currently pending packets with size equal to or larger than the size of the initial packet are eventually transmitted before the algorithm starts a smaller packet.

We start by some notations. We assume there are  $k$  distinct non-zero packet sizes denoted by  $\ell_i$  and ordered so that  $\ell_1 < \dots < \ell_k$ . For convenience, we define  $\ell_0 = 0$ . We say that the packet sizes are divisible if  $\ell_i$  divides  $\ell_{i+1}$  for all  $i = 1, \dots, k - 1$ . For a packet  $p$ , let  $\ell(p)$  denote the size of  $p$ . For a set of packets  $A$ , let  $\ell(A)$  denote the total size of all the packets in  $A$ .

During the run of an algorithm, at time  $t$ , a packet is pending if it is released before or at  $t$  and not started before  $t$  and still running. At time  $t$ , if no packet is running, the algorithm may start any pending packet. As a convention of our model, if a fault (jamming error) happens at time  $t$  and this is the completion time of a previously scheduled packet, this packet is considered completed. Also, at the fault time, the algorithm may start a new packet.

Let  $L_{ALG}(i, Y)$  denote the total size of packets of size  $\ell_i$  completed by an algorithm  $ALG$  during a

time interval  $Y$ . Similarly,  $L_{ALG}(\geq i, Y)$  (resp.  $L_{ALG}(< i, Y)$ ) denotes the total size of packets of size at least  $\ell_i$  (resp. less than  $\ell_i$ ) completed by an algorithm  $ALG$  during a time interval  $Y$ ; formally we define  $L_{ALG}(\geq i, Y) = \sum_{j=i}^k L_{ALG}(j, Y)$  and  $L_{ALG}(< i, Y) = \sum_{j=1}^{i-1} L_{ALG}(j, Y)$ . We use the notation  $L_{ALG}(Y)$  with a single parameter to denote the size  $L_{ALG}(\geq 1, Y)$  of packets of all sizes completed by  $ALG$  during  $Y$  and the notation  $L_{ALG}$  without parameters to denote the size of all packets of all sizes completed by  $ALG$  at any time.

By convention, the schedule starts at time 0 and ends at time  $T$ , which is a part of the instance unknown to an online algorithm until it is reached. (This is similar to the times of jamming errors, one can also alternatively say that after  $T$  the errors are so frequent that no packet is completed.) Algorithm  $ALG$  is called  $R$ -competitive, if there exists a constant  $C$ , possibly dependent on  $k$  and  $\ell_1, \dots, \ell_k$ , such that for any instance and its optimal schedule  $OPT$  we have  $L_{OPT}((0, T]) \leq R \cdot L_{ALG}((0, T]) + C$ . We remark that in our analyses we show only a crude bounds on  $C$ .

We denote the algorithm  $ALG$  with speedup  $s \geq 1$  by  $ALG(s)$ . The meaning is that in  $ALG(s)$ , packets of size  $L$  need time  $L/s$  to process. In the resource-augmentation variant, we are mainly interested in finding the smallest  $s$  such that  $ALG(s)$  is 1-competitive, compared to  $OPT = OPT(1)$  that runs at speed 1.

## 2.1 Algorithms *MAIN* and *DIV*

We divide the run of the algorithm into phases. Each phase starts by an invocation of Step (2) and ends either by a fault or by an invocation of Step (4). The periods of idle time, i.e., waiting in Step (1), do not belong to any phase. Throughout the algorithm,  $t$  denotes the current time. The time  $t_B$  denotes the last invocation of Step (2), which is the start of the current phase. We set  $\text{rel}(t) = s(t - t_B)$ . Since the algorithm does not insert unnecessary idle time,  $\text{rel}(t)$  denotes the amount of transmitted packets in the current phase. (Note that we use  $\text{rel}(t)$  only when there is no packet running at time  $t$ , so there is no partially executed packet.) Thus  $\text{rel}(t)$  can be thought as a measure of time relative to the start of the current phase (scaled by the speed of the algorithm). Note also that the algorithm can evaluate  $\text{rel}(t)$  without knowing the speedup, as it can simply observe the total size of the transmitted packets.

We now give a common description of the two algorithms, *MAIN* for general packet sizes and *DIV* for divisible sizes. The algorithms differ only in Step (3), where *DIV* enforces an additional *divisibility condition*. Let  $P^{<i}$  denote the set of pending packets of sizes  $\ell_1, \dots, \ell_{i-1}$  at any given time.

- (1) If no packet is available, stay idle until the next release time.
- (2) Let  $i$  be the maximal  $i \leq k$  such that there is a pending packet of size  $\ell_i$  and  $\ell(P^{<i}) < \ell_i$ . Schedule a packet of size  $\ell_i$ .
- (3) Choose the maximum  $i$  such that (i) there is a pending packet of size  $\ell_i$ , (ii)  $\ell_i \leq \text{rel}(t)$  and (iii) in case of *DIV*,  $\ell_i$  divides  $\text{rel}(t)$ . Schedule a packet of size  $\ell_i$ . Repeat Step (3) as long as such  $i$  exists.
- (4) If no packet satisfies the condition in Step (3), go to Step 1.

We first note that the algorithm is well-defined, i.e., that it is always able to choose a packet in Step (2) if it has any packets pending, and that if it succeeds in sending it, the length of thus started phase can be related to the total size of the packets completed in it.

**Lemma 2.1.** *Let  $ALG$  be either *MAIN* or *DIV*. In Step (2),  $ALG$  always chooses some packet if it has any pending. Moreover, if  $ALG$  completes the first packet in the phase, then  $L_{ALG(s)}((t_B, t_E]) > s(t_E - t_B)/2$ , where  $t_B$  denotes the start of the phase and  $t_E$  its end (by a fault or Step (4)).*

*Proof.* For the first property, note that the packet of the smallest size is eligible. For the second property, note that there is no idle time in the phase, and that only the last packet chosen by  $ALG$  in the phase may not complete due to a jam. By the condition in Step (3), the size of this jammed packet is no larger than the total size of all the packets  $ALG$  previously completed in this phase, which yields the bound.  $\square$

The following lemma shows a crucial property of the algorithm, namely that if packets of size  $\ell_i$  are pending, the algorithm schedules packets of size at least  $\ell_i$  most of the time. Its proof also explains the reasons behind our choice of the first packet in a phase in Step (2) of the algorithm. We give the statement and proof of the lemma for *MAIN*. For *DIV*, the first part of the lemma still holds, while the divisibility condition makes the proof a bit more tedious; on the other hand divisibility of packet sizes ensures that the bound in the second part of the lemma can be improved. We analyze *DIV* in Section 3.3.

**Lemma 2.2.** *Let  $u$  be a start of a phase in  $MAIN(s)$  and  $t = u + \ell_i/s$ .*

- (i) If a packet of size  $\ell_i$  is pending at time  $u$  and no fault occurs in  $(u, t)$ , then the phase does not end before  $t$ .
- (ii) Suppose that  $v > u$  is such that any time in  $[u, v)$  a packet of size  $\ell_i$  is pending and no fault occurs. Then the phase does not end in  $(u, v)$  and  $L_{MAIN(s)}(< i, (u, v]) < \ell_i + \ell_{i-1}$ . (Recall that  $\ell_0 = 0$ .)

*Proof.* (i) Suppose for a contradiction that the phase started at  $u$  ends at time  $t' < t$ . We have  $\text{rel}(t') < \text{rel}(t) = \ell_i$ . Let  $\ell_j$  be the smallest packet size among the packets pending at  $t'$ . As there is no fault, the reason for a new phase has to be that  $\text{rel}(t') < \ell_j$ , and thus Step (3) did not choose a packet to be scheduled. Also note that any packet scheduled before  $t'$  was completed. This implies, first, that there is a pending packet of size  $\ell_i$ , as there was one at time  $u$  and there was insufficient time to complete it, so  $j$  is well-defined and  $j \leq i$ . Second, all packets of sizes smaller than  $\ell_j$  pending at  $u$  were completed before  $t'$ , so their total size is at most  $\text{rel}(t') < \ell_j$ . However, this contradicts the fact that the phase started by a packet smaller than  $\ell_j$  at time  $u$ , as a pending packet of the smallest size equal to or larger than  $\ell_j$  satisfied the condition in Step (2) at time  $u$ . (Note that it is possible that no packet of size  $\ell_j$  is pending at  $u$ ; however, at least a packet of size  $\ell_i$  is pending.)

(ii) By (i), the phase that started at  $u$  does not end before time  $t$  if no fault happens. A packet of size  $\ell_i$  is always pending by the assumption of the lemma, and it is always a valid choice of a packet in Step (3) from time  $t$  on. Thus, the phase that started at  $u$  does not end in  $(u, v)$ , and moreover only packets of sizes at least  $\ell_i$  are started in  $[t, v)$ . It follows that packets of sizes smaller than  $\ell_i$  are started only before time  $t$  and their total size is thus less than  $\text{rel}(t) + \ell_{i-1} = \ell_i + \ell_{i-1}$ . The lemma follows.  $\square$

### 3 Local analysis and results

In this section we formulate a general method for analyzing our algorithms by comparing locally within each phase the size of “large” packets completed by the algorithm and by the adversary. This method simplifies a complicated induction used in [7]; we easily obtain the same upper bounds of 2 and 3 on competitiveness for divisible and unrestricted packet sizes respectively at no speedup, as well as several new results for the non-divisible cases. In Section 4, we use a more complex charging scheme to obtain our main result.

For the analysis, let  $ALG$  denote one of the algorithms  $MAIN$  and  $DIV$ , and let  $s \geq 1$  be the speedup. We fix an instance and its schedules for  $ALG(s)$  and  $OPT$ .

#### 3.1 Critical times and Master Theorem

The common scheme is the following. We carefully define a sequence of critical times  $C_k \leq C_{k-1} \leq \dots \leq C_1 \leq C_0$ , where  $C_0 = T$  is the end of the schedule, satisfying two properties: (1) the algorithm has completed almost all pending packets of size  $\ell_i$  released before  $C_i$  and (2) in  $(C_i, C_{i-1}]$ , a packet of size  $\ell_i$  is always pending. Properties (1) and (2) allow us to relate  $L_{OPT}(i, (0, C_i])$  and  $L_{OPT}(\geq i, (C_i, C_{i-1}])$  respectively to their “ $ALG$  counterparts”. As each packet completed by  $OPT$  belongs to exactly one of these sets, summing the bounds gives the desired results. These two facts together imply  $R$ -competitiveness of the algorithm for appropriate  $R$  and speed  $s$ .

We first define the notion of  $i$ -good times so that they satisfy property (1), and then choose the critical times among their suprema so that those satisfy property (2) as well.

**Definition 1.** Let  $ALG$  be one of the algorithms  $MAIN$  and  $DIV$ , and let  $s \geq 1$  be the speedup. For  $i = 1, \dots, k$ , time  $t$  is called  $i$ -good if one of the following conditions holds:

- (i) At time  $t$ , algorithm  $ALG(s)$  starts a new phase by scheduling a packet of size larger than  $\ell_i$ , or
- (ii) at time  $t$ , no packet of size  $\ell_i$  is pending for  $ALG(s)$ , or
- (iii)  $t = 0$ .

We define critical times  $C_0, C_1, \dots, C_k$  iteratively as follows:

- $C_0 = T$ , i.e., it is the end of the schedule.
- For  $i = 1, \dots, k$ ,  $C_i$  is the supremum of  $i$ -good times  $t$  such that  $t \leq C_{i-1}$ .

Note that all  $C_i$ 's are defined and  $C_i \geq 0$ , as time  $t = 0$  is  $i$ -good for all  $i$ .

The choice of  $C_i$  implies that each  $C_i$  is of one of the three types (the types are not disjoint):

- $C_i$  is  $i$ -good and a phase starts at  $C_i$  (this includes  $C_i = 0$ ),
- $C_i$  is  $i$ -good and  $C_i = C_{i-1}$ , or
- there exists a packet of size  $\ell_i$  pending at  $C_i$ , however, any such packet was released at  $C_i$ .

If the first two options do not apply, then the last one is the only remaining possibility (as otherwise some time in a non-empty interval  $(C_i, C_{i-1}]$  would be  $i$ -good); in this case,  $C_i$  is not  $i$ -good but only the supremum of  $i$ -good times.

First we bound the total size of packets of size  $\ell_i$  completed before  $C_i$ ; the proof actually only uses the fact that each  $C_i$  is the supremum of  $i$ -good times and justifies the definition above.

**Lemma 3.1.** *Let  $ALG$  be one of the algorithms  $MAIN$  and  $DIV$ , and let  $s \geq 1$  be the speedup. Then, for any  $i$ ,  $L_{OPT}(i, (0, C_i]) \leq L_{ALG(s)}(i, (0, C_i]) + \ell_k$ .*

*Proof.* If  $C_i$  is  $i$ -good and satisfies condition (i) in Definition 1, then by the description of Step (2) of the algorithms, the total size of pending packets of size  $\ell_i$  is less than the size of the scheduled packet, which is at most  $\ell_k$  and the lemma follows.

In all the remaining cases it holds that  $ALG(s)$  has completed all the jobs of size  $\ell_i$  released before  $C_i$ . Thus the inequality holds trivially even without the additive term.  $\square$

Our remaining goal is to bound  $L_{OPT}(\geq i, (C_i, C_{i-1}])$ . We divide  $(C_i, C_{i-1}]$  into  $i$ -segments by the faults. We prove the bounds separately for each  $i$ -segment. One important fact is that for the first  $i$ -segment we use only a loose bound, as we can use the additive constant. The critical part is then the bound for  $i$ -segments started by a fault, this part determines the bound and is different for each case. We summarize the general method by the following definition and master theorem.

**Definition 2.** *The interval  $(u, v]$  is called an initial  $i$ -segment if  $u = C_i$  and  $v$  is either  $C_{i-1}$  or the first time of a fault after  $u$ , whichever comes first.*

*The interval  $(u, v]$  is called a proper  $i$ -segment if  $u \in (C_i, C_{i-1})$  is a time of a fault and  $v$  is either  $C_{i-1}$  or the first time of a fault after  $u$ , whichever comes first.*

Note that there is no  $i$ -segment if  $C_{i-1} = C_i$ .

**Theorem 3.2** (Master Theorem). *Let  $ALG$  denote one of the algorithms  $MAIN$  and  $DIV$ , and let  $s \geq 1$  be the speedup. Suppose that  $R \geq 1$  satisfies that for each  $i = 1, \dots, k$  and each proper  $i$ -segment  $(u, v]$  with  $v - u \geq \ell_i$  we have*

$$(R - 1)L_{ALG(s)}((u, v]) + L_{ALG(s)}(\geq i, (u, v]) \geq L_{OPT}(\geq i, (u, v]). \quad (3.1)$$

Suppose furthermore that for the initial  $i$ -segment  $(u, v]$  we have

$$L_{ALG(s)}(\geq i, (u, v]) > s(v - u) - 4\ell_k. \quad (3.2)$$

Then  $ALG(s)$  is  $R$ -competitive.

*Proof.* First note that for a proper  $i$ -segment  $(u, v]$ ,  $u$  is a fault time. Thus if  $v - u < \ell_i$ , then  $L_{OPT}(\geq i, (u, v]) = 0$  and (3.1) is trivial. It follows that (3.1) holds even without the assumption  $v - u \geq \ell_i$ .

Now consider an initial  $i$ -segment  $(u, v]$ . We have  $L_{OPT}(\geq i, (u, v]) \leq \ell_k + v - u$ , as at most a single packet started before  $u$  can be completed. Combining this with (3.2) and using  $s \geq 1$ , we get  $L_{ALG(s)}(\geq i, (u, v]) > s(v - u) - 4\ell_k \geq v - u - 4\ell_k \geq L_{OPT}(\geq i, (u, v]) - 5\ell_k$ .

Summing this with (3.1) for all proper  $i$ -intervals and using  $R \geq 1$  we get

$$(R - 1)L_{ALG(s)}((C_i, C_{i-1}]) + L_{ALG(s)}(\geq i, (C_i, C_{i-1}]) + 5\ell_k > L_{OPT}(\geq i, (C_i, C_{i-1}]). \quad (3.3)$$

Note that for  $C_i = C_{i-1}$ , the equation (3.3) holds trivially.

To complete the proof of the theorem, note that each completed job in the optimum contributes to exactly one among the  $2k$  terms  $L_{OPT}(\geq i, (C_i, C_{i-1}])$  and  $L_{OPT}(i, (0, C_i])$ ; similarly for  $L_{ALG(s)}$ . Thus by summing both (3.3) and Lemma 3.1 for all  $i = 1, \dots, k$  we obtain

$$\begin{aligned} L_{OPT} &= \sum_{i=1}^k L_{OPT}(\geq i, (C_i, C_{i-1}]) + \sum_{i=1}^k L_{OPT}(i, (0, C_i]) \\ &< \sum_{i=1}^k (R - 1)L_{ALG(s)}((C_i, C_{i-1}]) + \sum_{i=1}^k (L_{ALG(s)}(\geq i, (C_i, C_{i-1}]) + 5\ell_k) \\ &\quad + \sum_{i=1}^k (L_{ALG(s)}(i, (0, C_i]) + \ell_k) \\ &\leq (R - 1)L_{ALG(s)} + L_{ALG(s)} + 6k\ell_k = R \cdot L_{ALG(s)} + 6k\ell_k. \end{aligned}$$

The theorem follows.  $\square$

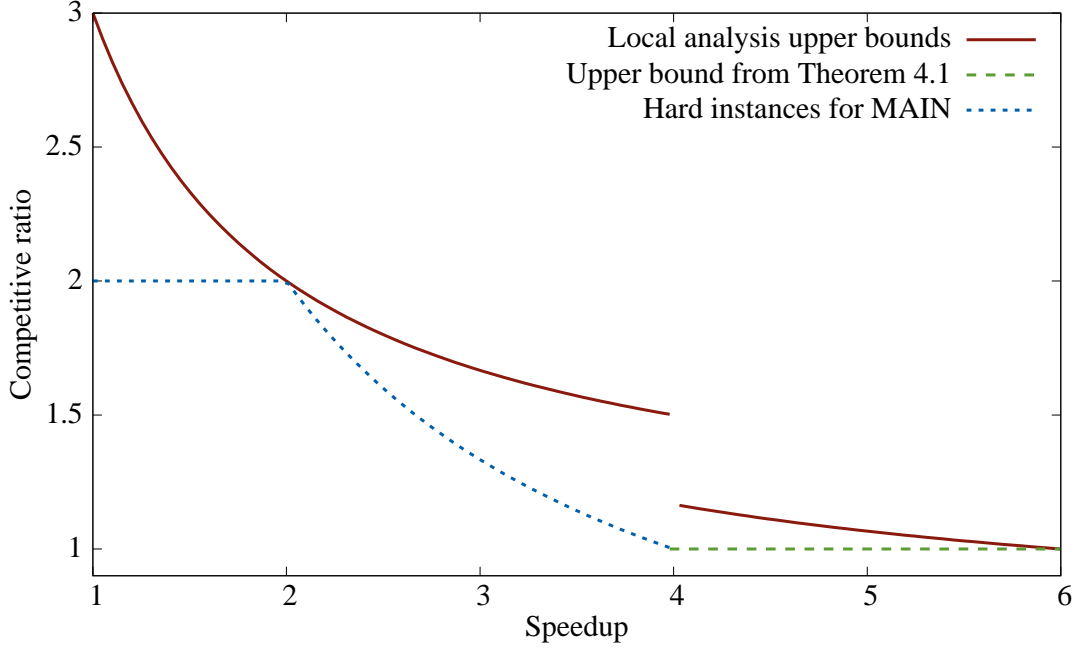


Figure 1: A graph of our upper and lower bounds on the competitive ratio of  $MAIN(s)$ . The lower bounds are by hard instances from Section 6.

## 3.2 Analysis of $MAIN$

The first part of the following lemma implies the condition (3.2) for the initial  $i$ -segments in all cases. The second part of the lemma is the base of the analysis of the proper  $i$ -segment, which is different in each situation.

**Lemma 3.3.** (i) If  $(u, v]$  is an initial  $i$ -segment then  $L_{MAIN(s)}(\geq i, (u, v]) > s(v - u) - 4\ell_k$ .  
(ii) If  $(u, v]$  is a proper  $i$ -segment and  $s(v - u) \geq \ell_i$  then  $L_{MAIN(s)}((u, v]) > s(v - u)/2$  and  $L_{MAIN(s)}(\geq i, (u, v]) > s(v - u)/2 - \ell_i - \ell_{i-1}$ . (Recall that  $\ell_0 = 0$ .)

*Proof.* (i) Let  $(u, v]$  be an initial  $i$ -segment. If the phase that starts at  $u$  or contains  $u$  ends before  $v$ , let  $u'$  be its end; otherwise let  $u' = u$ . We have  $u' \leq u + \ell_i/s$ , as otherwise any packet of size  $\ell_i$ , pending throughout the  $i$ -segment by definition, would be an eligible choice in Step (3) of the algorithm, and the phase would not end before  $v$ . Using Lemma 2.2(ii), we have  $L_{MAIN(s)}(< i, (u', v]) < \ell_i + \ell_{i-1} < 2\ell_k$ . Since at most one packet at the end of the segment is unfinished, we have  $L_{MAIN(s)}(\geq i, (u, v]) \geq L_{MAIN(s)}(\geq i, (u', v]) > s(v - u') - 3\ell_k > s(v - u) - 4\ell_k$ .

(ii) Let  $(u, v]$  be a proper  $i$ -segment. Thus  $u$  is a start of a phase that contains at least the whole interval  $[u, v]$  by Lemma 2.2(ii). By the definition of  $C_i$ ,  $u$  is not  $i$ -good, so the phase starts by a packet of size at most  $\ell_i$ . If  $v - u \geq \ell_i$  then the first packet finishes (as  $s \geq 1$ ) and thus  $L_{MAIN(s)}((u, v]) > s(v - u)/2$  by Lemma 2.1. The total size of completed packets smaller than  $\ell_i$  is at most  $\ell_i + \ell_{i-1}$  by Lemma 2.2(ii), and thus  $L_{MAIN(s)}(\geq i, (u, v]) > s(v - u)/2 - \ell_i - \ell_{i-1}$ .  $\square$

### 3.2.1 General packet sizes

The next theorem gives a tradeoff of the competitive ratio of  $MAIN(s)$  and the speedup  $s$  using our local analysis. In Theorem 4.1 we give a non-local analysis of  $MAIN(s)$  and show that the algorithm is 1-competitive for  $s \geq 4$ . Here we give a weaker result for  $s \geq 4$  that reflects the limits of the local analysis. However, for  $s = 1$  our local analysis is tight as already the lower bound from [1] shows that no algorithm is better than 3-competitive (for packet sizes 1 and  $2 - \epsilon$ ). See Figure 1 for an illustration of our upper and lower bounds on the competitive ratio of  $MAIN(s)$ .

**Theorem 3.4.**  $MAIN(s)$  is  $R_s$ -competitive where:

$$\begin{aligned} R_s &= 1 + 2/s \text{ for } s \in [1, 4), \\ R_s &= 2/3 + 2/s \text{ for } s \in [4, 6), \text{ and} \\ R_s &= 1 \text{ for } s \geq 6. \end{aligned}$$

*Proof.* Lemma 3.3(i) implies the condition (3.2) for the initial  $i$ -segments. We now prove (3.1) for any proper  $i$ -segment  $(u, v]$  with  $v - u \geq \ell_i$  and appropriate  $R$ . The bound then follows by the Master Theorem.

Since there is a fault at time  $u$ , we have  $L_{OPT}(\geq i, (u, v]) \leq v - u$ .

For  $s \geq 6$ , Lemma 3.3(ii) implies

$$L_{MAIN(s)}(\geq i, (u, v]) > s(v - u)/2 - 2\ell_i \geq 3(v - u) - 2(v - u) = v - u \geq L_{OPT}(\geq i, (u, v]),$$

which is (3.1) for  $R = 1$ .

For  $s \in [4, 6)$ , we first note that Lemma 3.3(ii) implies

$$\left(\frac{2}{s} - \frac{1}{3}\right) \cdot L_{MAIN(s)}((u, v]) > \left(1 - \frac{s}{6}\right)(v - u),$$

and thus to prove (3.1) for  $R = 2/3 + 2/s$ , it suffices to prove that

$$L_{MAIN(s)}(\geq i, (u, v]) > \frac{s}{6}(v - u),$$

as clearly  $v - u \geq L_{OPT}(\geq i, (u, v])$ . The remaining inequality again follows from Lemma 3.3(ii), but we need to consider two cases:

If  $(v - u) \geq \frac{6}{s}\ell_i$ , then

$$L_{MAIN(s)}(\geq i, (u, v]) > \frac{s}{2}(v - u) - 2\ell_i \geq \frac{s}{2}(v - u) - \frac{s}{3}(v - u) = \frac{s}{6}(v - u).$$

On the other hand, if  $(v - u) < \frac{6}{s}\ell_i$ , then using  $s \geq 4$  as well,

$$L_{MAIN(s)}(\geq i, (u, v]) > \frac{s}{2}(v - u) - 2\ell_i \geq 0,$$

which by divisibility implies

$$L_{MAIN(s)}(\geq i, (u, v]) \geq \ell_i > \frac{s}{6}(v - u).$$

For  $s \in [1, 4)$ , by Lemma 3.3(ii) we get  $(2/s) \cdot L_{MAIN(s)}((u, v]) > v - u \geq L_{OPT}(\geq i, (u, v])$ , which implies (3.1) for  $R = 1 + 2/s$ .  $\square$

### 3.2.2 Divisible and well-separated packet sizes

We can obtain better bounds on the speedup necessary for 1-competitiveness if the packet sizes are sufficiently different. We call the packet sizes  $\ell_1, \dots, \ell_k$   $\alpha$ -separated if  $\ell_i \geq \alpha\ell_{i-1}$  holds for  $i = 2, \dots, k$ . We start by analysis of  $MAIN(s)$  for the case of divisible packet sizes, which is a special case of 2-separated packet sizes. Note that by examples in Section 6 the bound of 2.5 is tight for our algorithm.

**Theorem 3.5.** *If the packet sizes are divisible, then  $MAIN(s)$  is 1-competitive for  $s \geq 2.5$ .*

*Proof.* Lemma 3.3(i) implies (3.2). We now prove (3.1) for any proper  $i$ -segment  $(u, v]$  with  $v - u \geq \ell_i$  and  $R = 1$ . The bound then follows by the Master Theorem. Since there is a fault at time  $u$ , we have  $L_{OPT}(\geq i, (u, v]) \leq v - u$ .

By divisibility we have  $L_{OPT}(\geq i, (u, v]) = n\ell_i$  for some positive integer  $n$ . We distinguish two cases based on the size of the last packet started by  $MAIN$  in the  $i$ -segment  $(u, v]$ , which is possibly unfinished due to a fault at  $v$ .

If the unfinished packet has size at most  $n\ell_i$ , then  $L_{MAIN(s)}(\geq i, (u, v]) > 5(v - u)/2 - 3\ell_i/2 - n\ell_i \geq 5n\ell_i/2 - 3\ell_i/2 - n\ell_i \geq (n - 1)\ell_i$  by Lemma 2.1 and Lemma 2.2(ii). Divisibility now implies  $L_{MAIN(s)}(\geq i, (u, v]) \geq n\ell_i = L_{OPT}(\geq i, (u, v])$ .

Otherwise by divisibility the size of the unfinished packet is at least  $(n + 1)\ell_i$  and the size of the completed packets is larger by the condition in Step (3) of the algorithm; here we also use the fact that  $MAIN(s)$  completes the packet started at  $u$ , as its size is at most  $\ell_i \leq v - u$ . Thus  $L_{MAIN(s)}(\geq i, (u, v]) > (n + 1)\ell_i - 3\ell_i/2 \geq (n - 1/2)\ell_i$ . Divisibility again implies  $L_{MAIN(s)}(\geq i, (u, v]) \geq n\ell_i = L_{OPT}(\geq i, (u, v])$ .

The theorem now follows by the Master Theorem.  $\square$



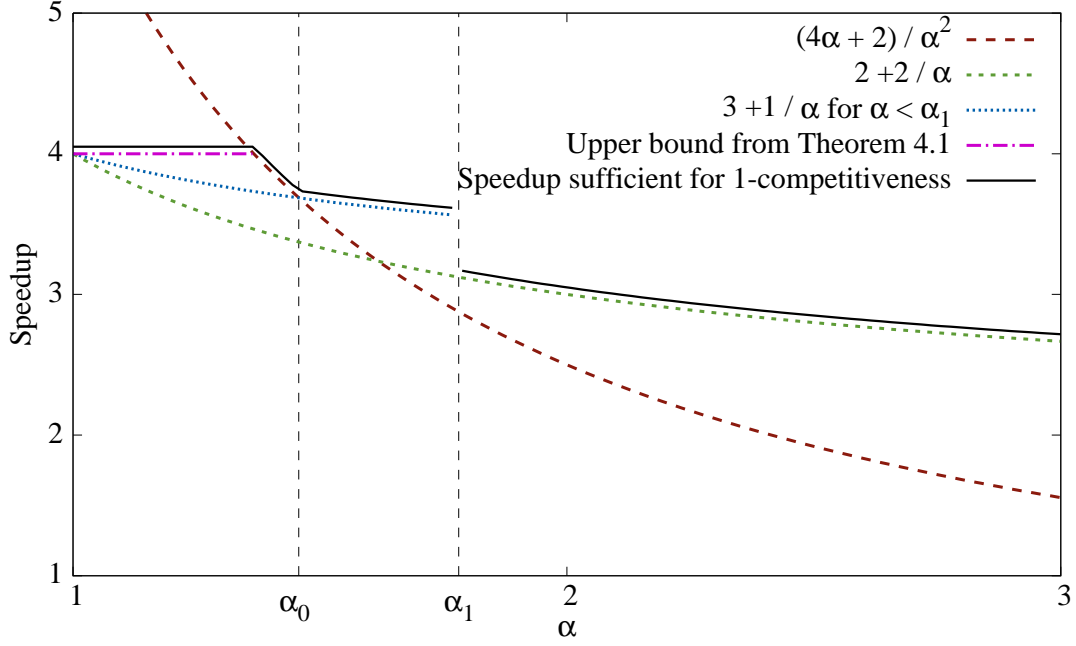


Figure 2: A graph of  $S_\alpha$  and the bounds on the speedup that we use in Theorem 3.6.

Next, we show that for  $\alpha$ -separated packet sizes,  $MAIN(S_\alpha)$  is 1-competitive for the following  $S_\alpha$ . We define

$$\alpha_0 = \frac{1}{2} + \frac{1}{6}\sqrt{33} \approx 1.46, \text{ which is the positive root of } 3\alpha^2 - 3\alpha - 2.$$

$$\alpha_1 = \frac{3 + \sqrt{17}}{4} \approx 1.78, \text{ which is the positive root of } 2\alpha^2 - 3\alpha - 1.$$

$$S_\alpha = \begin{cases} \frac{4\alpha + 2}{\alpha^2} & \text{for } \alpha \in [1, \alpha_0], \\ 3 + \frac{1}{\alpha} & \text{for } \alpha \in [\alpha_0, \alpha_1], \text{ and} \\ 2 + \frac{2}{\alpha} & \text{for } \alpha \geq \alpha_1. \end{cases}$$

The value of  $\alpha_0$  is chosen as the point where  $(4\alpha + 2)/\alpha^2 = 2 + 2/\alpha$ . The value of  $\alpha_1$  is chosen as the point from which the argument in case (viii) of the proof below works, which allows for a better result for  $\alpha \geq \alpha_1$ . If  $s \geq S_\alpha$  then  $s \geq (4\alpha + 2)/\alpha^2$  and  $s \geq 2 + 2/\alpha$  for all  $\alpha$  and also  $s \geq 3 + 1/\alpha$  for  $\alpha \leq \alpha_1$ ; these facts follow from inspection of the functions and are useful for the analysis. See Figure 2 for a graph of  $S_\alpha$  and all the bounds on it that we use.

Note that  $S_\alpha$  is decreasing in  $\alpha$ , with a single discontinuity at  $\alpha_1$ . We have  $S_1 = 6$ , matching Theorem 3.4. We have  $S_2 = 3$ , i.e.,  $MAIN(3)$  is 1-competitive for 2-separated packet sizes, which includes the case of divisible packet sizes studied above. The limit of  $S_\alpha$  for  $\alpha \rightarrow +\infty$  is 2. For  $\alpha < (1 + \sqrt{3})/2 \approx 1.366$ , we get  $S_\alpha > 4$ , while Theorem 4.1 shows that  $MAIN(s)$  is 1-competitive for  $s \geq 4$ ; the weaker result of Theorem 3.6 below reflects the limits of the local analysis.

**Theorem 3.6.** *Let  $\alpha > 1$ . If the packet sizes are  $\alpha$ -separated, then  $MAIN(s)$  is 1-competitive for any  $s \geq S_\alpha$ .*

*Proof of Theorem 3.6.* Lemma 3.3(i) implies (3.2). We now prove for any proper  $i$ -segment  $(u, v]$  with  $v - u \geq \ell_i$  that

$$L_{MAIN(s)}(\geq i, (u, v]) \geq L_{OPT}(\geq i, (u, v]), \quad (3.4)$$

which is (3.1) for  $R = 1$ . The bound then follows by the Master Theorem.

Let  $X = L_{OPT}(\geq i, (u, v])$ . Note that  $X \leq v - u$ .

Lemma 3.3(ii) together with  $\ell_{i-1} \leq \ell_i/\alpha$  gives  $L_{MAIN(s)}(\geq i, (u, v]) > M$  for  $M = sX/2 - (1 + 1/\alpha)\ell_i$ .

As in the case of divisible packet sizes, we use the fact that both  $X$  and  $L_{MAIN(s)}(\geq i, (u, v])$  are sums of some packet sizes  $\ell_j$ ,  $j \geq i$ , and thus only some of the values are possible. Here the situation is more complicated, as for example  $\ell_{i+1}$ ,  $\ell_{i+2}$ ,  $2\ell_i$ ,  $\ell_i + \ell_{i+1}$  are possible values, but their ordering may vary.

We distinguish several cases based on  $X$  and  $\alpha$ . We note in advance that the first five cases suffice for  $\alpha < \alpha_1$ ; only after completing the proof for  $\alpha < \alpha_1$ , we analyze the additional cases needed for  $\alpha \geq \alpha_1$ .

**Case (i):**  $X = 0$ . Then (3.4) is trivial.

**Case (ii):**  $X = \ell_i$ . Using  $s \geq 2 + 2/\alpha$ , we obtain  $M \geq 0$ . Thus  $L_{MAIN(s)}(\geq i, (u, v]) > M \geq 0$  which implies  $L_{MAIN(s)}(\geq i, (u, v]) \geq \ell_i = X$  and (3.4) holds.

**Case (iii):**  $X = \ell_{i+1} \leq 2\ell_i$ . Using  $s \geq (4\alpha + 2)/\alpha^2$  and  $X = \ell_{i+1} \geq \alpha\ell_i$ , we obtain  $M \geq (2 + 1/\alpha)\ell_i - (1 + 1/\alpha)\ell_i = \ell_i$ . Thus  $L_{MAIN(s)}(\geq i, (u, v]) > \ell_i$  which together with  $\ell_{i+1} \leq 2\ell_i$  implies  $L_{MAIN(s)}(\geq i, (u, v]) \geq \ell_{i+1} = X$  and (3.4) holds.

**Case (iv):**  $X \geq \alpha^2\ell_i$ . (Note that this includes all cases when a packet of size at least  $\ell_{i+2}$  contributes to  $X$ .) We first show that  $s \geq 2(1 + 1/\alpha^2 + 1/\alpha^3)$  by straightforward calculations with the golden ratio  $\phi$ : If  $\alpha \leq \phi$ , we have  $1/\alpha + 1/\alpha^2 - 1 \geq 0$  and thus  $s \geq (4\alpha + 2)/\alpha^2 = 2(2/\alpha + 1/\alpha^2) \geq 2(1 + 1/\alpha^2 + 1/\alpha^3)$ , as  $2/\alpha \geq 1 + 1/\alpha^3$  is equivalent to  $(\alpha - 1)(\alpha^2 - \alpha - 1) \leq 0$ , which holds for  $\alpha \in (1, \phi)$ . If on the other hand  $\alpha \geq \phi$ , then  $s \geq 2(1 + 1/\alpha) \geq 2(1 + 1/\alpha^2 + 1/\alpha^3)$ , as  $1/\alpha \geq 1/\alpha^2 + 1/\alpha^3$  holds for  $\alpha \geq \phi$ . Now we obtain

$$\begin{aligned} M - X &\geq \left(\frac{s}{2} - 1\right)X - \left(1 + \frac{1}{\alpha}\right)\ell_i \geq \left(1 + \frac{1}{\alpha^2} + \frac{1}{\alpha^3} - 1\right)X - \left(1 + \frac{1}{\alpha}\right)\ell_i \\ &\geq \left(\frac{1}{\alpha^2} + \frac{1}{\alpha^3}\right)\alpha^2\ell_i - \left(1 + \frac{1}{\alpha}\right)\ell_i = 0, \end{aligned}$$

and (3.4) holds.

**Case (v):**  $X \geq 2\ell_i$  and  $\alpha < \alpha_1$ . (Note that, as long as  $\alpha < \alpha_1$ , this includes all cases when at least two packets contribute to  $X$ .) Using  $s \geq 1 + 3/\alpha$  we obtain

$$M - X \geq \left(\frac{1}{2}\left(3 + \frac{1}{\alpha}\right) - 1\right)X - (1 + 1/\alpha)\ell_i \geq \frac{1}{2}\left(1 + \frac{1}{\alpha}\right)2\ell_i - (1 + 1/\alpha)\ell_i = 0,$$

and (3.4) holds.

**Proof for  $\alpha < \alpha_1$ :** We now observe that for  $\alpha < \alpha_1$ , we have exhausted all the possible values of  $X$ . Indeed, if (v) does not apply, then at most a single packet contributes to  $X$ , and one of the cases (i)-(iv) applies, as (iv) covers the case when  $X \geq \ell_{i+2}$ . Thus (3.4) holds and the proof is complete.

**Proof for  $\alpha \geq \alpha_1$ :** We now analyze the remaining cases for  $\alpha \geq \alpha_1$ .

**Case (vi):**  $X \geq (\alpha + 1)\ell_i$ . (Note that this includes all cases when two packets not both of size  $\ell_i$  contribute to  $X$ .) Using  $s \geq 2 + 2/\alpha$  we obtain  $M - X \geq (1 + 1/\alpha - 1)X - (1 + 1/\alpha)\ell_i \geq 0$  and (3.4) holds.

**Case (vii):**  $X = n \cdot \ell_i < (\alpha + 1)\ell_i$  for some  $n = 2, 3, \dots$ . Since  $\alpha > \alpha_1$ , we have  $\ell_{i+1} > \ell_i + \ell_{i-1}$ . This implies that the first packet of size at least  $\ell_i$  that is scheduled in the phase has size equal to  $\ell_i$  by the condition in Step (3) of the algorithm. Thus, if also a packet of size larger than  $\ell_i$  contributes to  $L_{MAIN(s)}(\geq i, (u, v])$ , we have  $L_{MAIN(s)}(\geq i, (u, v]) \geq \ell_{i+1} + \ell_i \geq (\alpha + 1)\ell_i > X$  by the case condition and (3.4) holds. Otherwise  $L_{MAIN(s)}(\geq i, (u, v])$  is a multiple of  $\ell_i$ . Using  $s \geq 2 + 2/\alpha$ , we obtain  $M \geq (n - 1)(1 + 1/\alpha)\ell_i > (n - 1)\ell_i$ . This, together with divisibility by  $\ell_i$  implies  $L_{MAIN(s)}(\geq i, (u, v]) \geq n \cdot \ell_i = X$  and (3.4) holds again.

**Case (viii):**  $X = \ell_{i+1} > 2\ell_i$ . We distinguish two subcases depending on the size of the unfinished packet of  $MAIN(s)$  in this phase.

If the unfinished packet has size at most  $\ell_{i+1}$ , the size of the completed packets is bounded by  $L_{MAIN(s)}((u, v]) > sX - \ell_{i+1} = (s - 1)\ell_{i+1} > (1 + 2/\alpha)\ell_{i+1}$ , using  $s \geq 2 + 2/\alpha$ . Since the total size of packets smaller than  $\ell_i$  is less than  $(1 + 1/\alpha)\ell_i$  by Lemma 2.2(ii), we obtain  $L_{MAIN(s)}(\geq i, (u, v]) - X > 2\ell_{i+1}/\alpha - (1 + 1/\alpha)\ell_i > 0$ , as  $\ell_{i+1}/\alpha \geq \ell_i$ . Thus (3.4) holds.

Otherwise the unfinished packet has size at least  $\ell_{i+2}$  and, by Step (3) of the algorithm, also  $L_{MAIN(s)}((u, v]) > \ell_{i+2}$ . We have  $\ell_{i+2} \geq \alpha\ell_{i+1}$  and by the case condition  $\ell_{i+1} > 2\ell_i$  we obtain  $L_{MAIN(s)}(\geq i, (u, v]) - X > (\alpha - 1)\ell_{i+1} - (1 + 1/\alpha)\ell_i > 2(\alpha - 1)\ell_i - (1 + 1/\alpha)\ell_i > 0$ , as the definition of  $\alpha_1$  implies that  $2(\alpha - 1) \geq 1 + 1/\alpha$  for  $\alpha \geq \alpha_1$ . Thus (3.4) holds.

We now observe that we have exhausted all the possible values of  $X$  for  $\alpha \geq \alpha_1$ . Indeed, if at least two packets contribute to  $X$ , either (vi) or (vii) applies. Otherwise, at most a single packet contributes to  $X$ , and one of the cases (i)-(iv) or (viii) applies, as (iv) covers the case when  $X \geq \ell_{i+2}$ . Thus (3.4) holds and the proof is complete.  $\square$

### 3.3 Analysis of *DIV*

In this section we analyze *DIV*, giving a simple proof that it matches the performance of the algorithms from [7] on divisible instances.

Throughout the section we assume that the packet sizes are divisible. First we prove a lemma stating some properties of the schedule implied by divisibility.

We start by a few observations on divisibility that lead to a strengthening of Lemma 2.2.

- Lemma 3.7.** (i) *If  $DIV$  starts or completes a packet of size  $\ell_i$  at time  $t$ , then  $\ell_i$  divides  $\text{rel}(t)$ .*  
(ii) *Let  $t$  be a time with  $\text{rel}(t)$  divisible by  $\ell_i$  and  $\text{rel}(t) > 0$ . If a packet of size  $\ell_i$  is pending at time  $t$ , then  $DIV$  starts or continues running a packet of size at least  $\ell_i$  at time  $t$ .*  
(iii) *If at the beginning of phase at time  $u$  a packet of size  $\ell_i$  is pending and no fault occurs before time  $t = u + \ell_i/s$ , then the phase does not end before  $t$ .*

*Proof.* (i) follows trivially from the description of the algorithm.

(ii): If *DIV* continues running some packet at  $t$ , it cannot be a packet smaller than  $\ell_i$  by (i) and the claim follows. If *DIV* starts a new packet, then a packet of size  $\ell_i$  is pending by the assumption. Furthermore, it satisfies all the conditions from Step 3 of the algorithm, as  $\text{rel}(t)$  is divisible by  $\ell_i$  and  $\text{rel}(t) \geq \ell_i$  (from  $\text{rel}(t) > 0$  and divisibility).

(iii): We proceed by induction on  $i$ . Assume that no fault happens before  $t$ . If the phase starts by a packet of size at least  $\ell_i$ , the claim holds trivially, as the packet is not completed before  $t$ . This also proves the base of the induction for  $i = 1$ .

It remains to handle the case when the phase starts by a packet smaller than  $\ell_i$ . Let  $A$  be the set of all packets of size smaller than  $\ell_i$  pending at time  $u$ . By the Step (2) of the algorithm,  $\ell(A) \geq \ell_i$ . We show that all packets of  $A$  are completed if no fault happens, which implies that the phase does not end before  $t$ .

Let  $j$  be such that  $\ell_j$  is the maximum size of a packet in  $A$ ; note that  $j$  exists, as the phase starts by a packet smaller than  $\ell_i$ . By the induction assumption, the phase does not end before time  $t' = u + \ell_j/s$ . From time  $t'$  on, the conditions in Step (3) guarantee that the remaining packets from  $A$  are processed from the largest ones, possibly interleaved with some of the newly arriving packets of larger sizes, as  $\text{rel}(\tau)$  for the current time  $\tau$  is always divisible by the size of the largest pending packet from  $A$ . This shows that the phase cannot end before all packets from  $A$  are completed, if no fault happens.  $\square$

Now we prove a stronger analogue of Lemma 3.3.

- Lemma 3.8.** (i) *If  $(u, v]$  is an initial  $i$ -segment then  $L_{DIV(s)}(\geq i, (u, v]) > s(v - u) - 3\ell_k$ .*  
(ii) *If  $(u, v]$  is a proper  $i$ -segment and  $s(v - u) \geq \ell_i$  then  $L_{DIV(s)}(\geq i, (u, v]) > s(v - u)/2 - \ell_i$ ; furthermore  $L_{DIV(s)}((u, v]) > s(v - u)/2$  and  $L_{DIV(s)}((u, v])$  is divisible by  $\ell_i$ .*

*Proof.* Suppose that time  $t \in [u, v)$  satisfies that  $\text{rel}(t)$  divisible by  $\ell_i$  and  $\text{rel}(t) > 0$ . Then observe that Lemma 3.7(ii) together with the assumption that a packet of size  $\ell_i$  is always pending in  $[u, v)$  implies that from time  $t$  on only packets of size at least  $\ell_i$  are scheduled, and thus the current phase does not end before  $v$ .

For a proper  $i$ -segment  $(u, v]$ , the previous observation for  $t = u + \ell_i/s$  immediately implies (ii): Observe that  $t \leq v$  by the assumption of (ii). Now  $L_{DIV(s)}(< i, (u, v])$  is either equal to 0 (if the phase starts by a packet of size  $\ell_i$  at time  $u$ ) or equal to  $\ell_i$  (if the phase starts by a smaller packet). In both cases  $\ell_i$  divides  $L_{DIV(s)}(< i, (u, v])$  and thus also  $L_{DIV(s)}((u, v])$ . As in the analysis of *MAIN*, the total size of completed packets is more than  $s(v - u)/2$  and (ii) follows.

For an initial  $i$ -segment  $(u, v]$  we first observe that the claim is trivial if  $s(v - u) \leq 2\ell_i$ . So we may assume that  $u + 2\ell_i/s \leq v$ . Now we distinguish two cases:

1. The phase of  $u$  ends at some time  $u' \leq u + \ell_i/s$ : Then, by Lemma 3.7(iii) and the initial observation, the phase that immediately follows the one of  $u$  does not end in  $(u', v)$  and from time  $u' + \ell_i/s$  on, only packets of size at least  $\ell_i$  are scheduled. Thus  $L_{DIV(s)}(< i, (u, v]) \leq 2\ell_i$ .
2. The phase of  $u$  does not end by time  $u + \ell_i/s$ : Thus there exists  $t \in (u, u + \ell_i/s]$  such that  $\ell_i$  divides  $\text{rel}(t)$  and also  $\text{rel}(t) > 0$  as  $t > u$ . Using the initial observation for this  $t$  we obtain that the phase does not end in  $(u, v)$  and from time  $t$  on only packets of size at least  $\ell_i$  are scheduled. Thus  $L_{DIV(s)}(< i, (u, v]) \leq \ell_i$ .

In both cases  $L_{DIV(s)}(< i, (u, v]) \leq 2\ell_i$ , furthermore only a single packet is possibly unfinished at time  $v$ . Thus  $L_{DIV(s)}(\geq i, (u, v]) > s(v - u) - 2\ell_i - \ell_k$  and (i) follows.  $\square$

**Theorem 3.9.** *Let the packet sizes be divisible. Then  $DIV(1)$  is 2-competitive. Also, for any speed  $s \geq 2$ ,  $DIV(s)$  is 1-competitive.*

*Proof.* Lemma 3.8(i) implies (3.2). We now prove (3.1) for any proper  $i$ -segment  $(u, v]$  with  $v - u \geq \ell_i$  and appropriate  $R$ . The theorem then follows by the Master Theorem.

Since  $u$  is a time of a fault, we have  $L_{OPT}(\geq i, (u, v]) \leq v - u$ . If  $L_{OPT}(\geq i, (u, v]) = 0$ , (3.1) is trivial. Otherwise  $L_{OPT}(\geq i, (u, v]) \geq \ell_i$ , thus  $v - u \geq \ell_i \geq \ell_i/s$  and the assumption of Lemma 3.8(ii) holds.

For  $s \geq 2$ , Lemma 3.8(ii) implies

$$L_{DIV(s)}(\geq i, (u, v]) > s(v - u)/2 - \ell_i \geq v - u - \ell_i \geq L_{OPT}(\geq i, (u, v]) - \ell_i.$$

Since both  $L_{DIV(s)}(\geq i, (u, v])$  and  $L_{OPT}(\geq i, (u, v])$  are divisible by  $\ell_i$ , this implies  $L_{DIV(s)}(\geq i, (u, v]) \geq L_{OPT}(\geq i, (u, v])$ , i.e., (3.1) holds for  $R = 1$ .

For  $s = 1$ , Lemma 3.8(ii) implies

$$L_{DIV}((u, v]) + L_{DIV}(\geq i, (u, v]) > (v - u)/2 + (v - u)/2 - \ell_i \geq v - u - \ell_i \geq L_{OPT}(\geq i, (u, v]) - \ell_i.$$

Since  $L_{DIV}((u, v])$ ,  $L_{DIV}(\geq i, (u, v])$ , and  $L_{OPT}(\geq i, (u, v])$  are all divisible by  $\ell_i$ , this implies  $L_{DIV}((u, v]) + L_{DIV}(\geq i, (u, v]) \geq L_{OPT}(\geq i, (u, v])$ , i.e., (3.1) holds for  $R = 1$ .  $\square$

## 4 MAIN with speed 4

In this section we prove that speed 4 is sufficient for *MAIN* to be 1-competitive. Examples in Section 6 show that speed 4 is also necessary for our algorithm.

**Theorem 4.1.** *MAIN*( $s$ ) is 1-competitive for  $s \geq 4$ .

**Intuition.** For  $s \geq 4$  we have that if at the start of a phase *MAIN*( $s$ ) has a packet of size  $\ell_i$  pending and the phase has length at least  $\ell_i$ , then *MAIN*( $s$ ) completes a packet of size at least  $\ell_i$ . To show this, assume that the phase starts at time  $t$ . Then the first packet  $p$  of size at least  $\ell_i$  is started before time  $t + 2\ell_i/s$  by Lemma 2.2(ii) and by the condition in Step (3) it has size smaller than  $2\ell_i$ . Thus it completes before time  $t + 4\ell_i/s \leq t + \ell_i$ , which is before the end of the phase. This property does not hold for  $s < 4$ . It is important in our proof, as it shows that if the optimal schedule completes a job of some size, and such job is pending for *MAIN*( $s$ ), then *MAIN*( $s$ ) completes a job of the same size or larger. However, this is not sufficient to complete the proof by a local (phase-by-phase) analysis similar to the previous section, as the next example shows.

Assume that at the beginning, we release  $N$  packets of size 1,  $N$  packets of size  $1.5 - 2\varepsilon$ , one packet of size  $3 - 2\varepsilon$  and a sufficient number of packets of size  $1 - \varepsilon$ , for a small  $\varepsilon > 0$ . Our focus is on packets of size at least 1. Supposing  $s = 4$  we have the following phases:

- First, there are  $N$  phases of length 1. In each phase the optimum completes a packet of size 1, while among packets of size at least 1, *MAIN*( $s$ ) completes a packet of size  $1.5 - 2\varepsilon$ , as it starts packets of sizes  $1 - \varepsilon, 1 - \varepsilon, 1.5 - 2\varepsilon, 3 - 2\varepsilon$ , in this order, and the last packet is jammed.
- Then there are  $N$  phases of length  $1.5 - 2\varepsilon$  where the optimum completes a packet of size  $1.5 - 2\varepsilon$  while among packets of size at least 1, the algorithm completes only a single packet of size 1, as it starts packets of sizes  $1 - \varepsilon, 1 - \varepsilon, 1, 3 - 2\varepsilon$ , in this order. The last packet is jammed, since for  $s = 4$  the phase must have length at least  $1.5 - \varepsilon$  to complete it.

In phases of the second type, the algorithm does not complete more (in terms of total size) packets of size at least 1 than the optimum. Nevertheless, in our example, packets of size  $1.5 - 2\varepsilon$  were already finished by the algorithm, and this is a general rule. The novelty in our proof is a complex charging argument that exploits such subtle interaction between phases.

**Outline of the proof.** We define critical times  $C'_i$  similarly as before, but without the condition that they should be ordered, i.e.,  $C'_i$  is simply the supremum of  $i$ -good times; consequently, either  $C'_i \leq C'_{i-1}$  or  $C'_i > C'_{i-1}$  may hold.

Our plan is to define, for each  $i$ , charges between packets of size  $\ell_i$  up to time  $C'_i$ , since nearly each packet of size  $\ell_i$  completed by the adversary before  $C'_i$  must be completed by the algorithm before  $C'_i$ . Each such charge from an adversary's packet leads to an algorithm's packet of the same size and each algorithm's packet receives at most one such charge; we thus call these charges 1-to-1 charges.

We want to charge the remaining adversary's packets of size  $\ell_i$  (scheduled after  $C'_i$ ) to some algorithm's packets in the same phase. For technical reasons it is actually more convenient to partition the schedule not into phases but into blocks inbetween successive faults. A block can contain several

phases of the algorithm separated by an execution of Step (4); however, in the most important and tight part of the analysis the blocks coincide with phases.

After the critical time  $C'_i$ , packets of size  $\ell_i$  are always pending for the algorithm, and thus (as we observed above) the algorithm schedules a packet of size at least  $\ell_i$  in blocks in which the adversary completes a packet of size  $\ell_i$ . In the crucial lemma of the proof, based on these observations and their refinements, we show that we can assign remaining adversary's packets (i.e., those without 1-to-1 charges) to algorithm's packets in the same block so that for each algorithm's packet  $q$  the total size of packets assigned to it is at most  $\ell(q)$ . However, we cannot use this assignment directly to charge the remaining packets, as some of the algorithm's big packets may receive 1-to-1 charges, and in this case the analysis needs to handle the interaction of different blocks. This very issue can be seen even in our introductory example.

To deal with this, we process blocks in the order of time from the beginning to the end of the schedule, simultaneously completing the charging to the packets in the current block of  $MAIN(s)$  schedule and possibly modifying  $ADV$  (the schedule of the adversary) in the future blocks. In fact, in the assignment described above, we include not only the packets in  $ADV$  without 1-to-1 charge but also packets in  $ADV$  with a 1-to-1 charge to a later block. After creating the assignment, if we have a packet  $q$  in  $ALG$  that receives a 1-to-1 charge from a packet  $p$  in a later block of  $ADV$ , we remove  $p$  from  $ADV$  in that later block and replace it there by the packets assigned to  $q$  (that are guaranteed to be of smaller total size than  $p$ ). After these swaps, the 1-to-1 charges together with the assignment form a valid charging that charges the remaining not swapped packets in  $ADV$  in this block together with the removed packets from the later blocks in  $ADV$  to the packets of  $MAIN(s)$  in the current block. This charging is now independent of the other blocks, so we can continue with the next block.

Furthermore, to analyze blocks in which the algorithm does not complete any packet, we shift the targets of the 1-to-1 charges backward in time by accounting for the first few packets of each size completed by the adversary at the beginning of the schedule and after  $C'_i$  in the additive constant of the competitive ratio.

**Preliminaries.** We define the notions of blocks and (modified) critical times.

**Definition 3.** Let  $f_1, f_2, \dots, f_N$  be the times of faults. Let  $f_0 = 0$  and  $f_{N+1} = T$  is the end of schedule. Then the time interval  $(f_i, f_{i+1}]$ ,  $i = 0, \dots, N$ , is called a block.

**Definition 4.** For  $i = 1, \dots, k$ , the critical time  $C'_i$  is the supremum of  $i$ -good times  $t \in [0, T]$ , where  $T$  is the end of the schedule and  $i$ -good times are as defined in Definition 1.

All  $C'_i$ 's are defined, as  $t = 0$  is  $i$ -good for all  $i$ . Similarly to Section 3.1, each  $C'_i$  is of one of the following types: (i)  $C'_i$  starts a phase and a packet larger than  $\ell_i$  is started, (ii)  $C'_i = 0$ , (iii)  $C'_i = T$ , or (iv) just before time  $C'_i$  no packet of size  $\ell_i$  is pending but at time  $C'_i$  one or more packets of size  $\ell_i$  are pending; in this case  $C'_i$  is not  $i$ -good but only the supremum of  $i$ -good times. We observe that in each case, at time  $C'_i$  the total size of packets  $p$  of size  $\ell_i$  pending for  $MAIN(s)$  released before  $C'_i$  is less than  $\ell_k$ .

**1-to-1 charges and the additive constant.** Next we define the set of packets that contribute to the additive constant.

**Definition 5.** Let the set  $A$  contain for each  $i = 1, \dots, k$ :

- (i) the first  $\lceil 4\ell_k/\ell_i \rceil$  packets of size  $\ell_i$  completed by the adversary, and
- (ii) the first  $\lceil 4\ell_k/\ell_i \rceil$  packets of size  $\ell_i$  completed by the adversary after  $C'_i$ .

If there are not sufficiently many packets of size  $\ell_i$  completed by the adversary in (i) or (ii), we take all the packets in (i) or all the packets completed after  $C'_i$  in (ii), respectively.

For each  $i$ , we put into  $A$  packets of size  $\ell_i$  of total size at most  $10\ell_k$ . Thus we have  $\ell(A) = O(k\ell_k)$  which implies that packets in  $A$  can be counted in the additive constant.

We define 1-to-1 charges for packets of size  $\ell_i$  as follows. Let  $p_1, p_2, \dots, p_n$  be all the packets of size  $\ell_i$  started by the adversary before  $C'_i$  that are not in  $A$ . We claim that  $MAIN(s)$  completes at least  $n$  packets of size  $\ell_i$  before  $C'_i$ . Indeed, before time  $C'_i$  at least  $n + \lceil 4\ell_k/\ell_i \rceil$  packets of size  $\ell_i$  are started by the adversary and thus released; by the definition of  $C'_i$  at time  $C'_i$  fewer than  $\ell_k/\ell_i$  of them are pending for  $MAIN(s)$ , one may be running and the remaining ones must be completed. We now charge each  $p_m$  to the  $m$ th packet of size  $\ell_i$  completed by  $MAIN(s)$ .

We call a 1-to-1 charge starting and ending in the same block an *up charge*, a 1-to-1 charge from a block starting at  $u$  to a block ending at  $v' \leq u$  a *back charge*, and a 1-to-1 charge from a block ending at  $v$  to a block starting at  $u' \geq v$  a *forward charge*. A *charged packet* is a packet charged by a 1-to-1 charge. The definition of  $A$  implies the following two important properties.

**Lemma 4.2.** *Let  $p$  be a packet of size  $\ell_i$ , started by the adversary at time  $t$ , charged by a forward charge to a packet  $q$  started by  $MAIN(s)$  at time  $t'$ . Then at any time  $\tau \in [t - 3\ell_k, t')$ , more than  $\ell_k/\ell_i$  packets of size  $\ell_i$  are pending for  $MAIN(s)$ .*

*Proof.* Let  $m$  be the number of packets of size  $\ell_i$  that  $MAIN(s)$  completes before  $q$ . Then, by the definition of  $A$ , the adversary completes  $m + \lceil 4\ell_k/\ell_i \rceil$  packets of size  $\ell_i$  before  $p$ . As fewer than  $3\ell_k/\ell_i$  of these packets are started in  $(t - 3\ell_k, t]$ , the remaining more than  $m + \ell_k/\ell_i$  packets have been released before or at time  $t - 3\ell_k$ . As only  $m$  of them are completed by  $MAIN(s)$  before  $t'$ , the remaining more than  $\ell_k/\ell_i$  packets are pending at any time  $\tau \in [t - 3\ell_k, t')$ .  $\square$

**Lemma 4.3.** *Let  $p \notin A$  be a packet of size  $\ell_i$  started by the adversary at time  $t$  that is not charged. Then  $t - 4\ell_k \geq C'_i$  and thus at any  $\tau \geq t - 4\ell_k$ , a packet of size  $\ell_i$  is pending for  $MAIN(s)$ .*

*Proof.* Any packet of size  $\ell_i$  started before  $C'_i + 4\ell_k$  is either charged or put in  $A$ , thus  $t - 4\ell_k \geq C'_i$ . After  $C'_i$ , a packet of size  $\ell_i$  is pending by the definition of  $C'_i$ .  $\square$

**Processing blocks.** Initially, let  $ADV$  be an optimal (adversary) schedule. First we remove all packets in  $A$  from  $ADV$ . Then we process blocks one by one in the order of time. When we process a block, we modify  $ADV$  so that we (i) remove some packets from  $ADV$ , so that the total size of removed packets is at most the total size of packets completed by  $MAIN(s)$  in this block, and (ii) reschedule any remaining packet in  $ADV$  in this block to one of the later blocks, so that the schedule of remaining packets remains feasible. Summing over all intervals, (i) guarantees that  $MAIN(s)$  is 1-competitive with an additive constant  $\ell(A)$ .

When we reschedule a packet in  $ADV$ , we keep the packet's 1-to-1 charge (if it has one), however, its type may change due to rescheduling. Since we are moving packets only to later times only, the release times are automatically respected. Also it follows that we can apply Lemmata 4.2 and 4.3 even to  $ADV$  after rescheduling.

After processing of a block, there will remain no charges to or from it. For the charges from the block this is automatic, as there remain no packets to be charged. For the charges to the block, this is guaranteed as in the process we remove from  $ADV$  all the packets in later blocks charged by back charges to the current block.

From now on, let  $(u, v]$  be the current block that we are processing; all previous blocks ending at  $v' \leq u$  are processed. As there are no charges to the previous blocks, any packet scheduled in  $ADV$  in  $(u, v]$  is charged by an up charge or a forward charge, or else it is not charged at all. We distinguish two main cases of the proof, depending on whether  $MAIN(s)$  finishes any packet in the current block.

**Main case 1: Empty block.** The algorithm does not finish any packet in  $(u, v]$ . We claim that  $ADV$  does not finish any packet. The processing of the block is then trivial.

For a contradiction, assume that  $ADV$  starts a packet  $p$  of size  $\ell_i$  at time  $t$  and completes it. The packet  $p$  cannot be charged by an up charge, as  $MAIN(s)$  completes no packet in this block. Thus  $p$  is either charged by a forward charge or not charged. Lemma 4.2 or 4.3 implies that at time  $t$  some packet of size  $\ell_i$  is pending for  $MAIN(s)$ .

Since  $MAIN$  does not idle unnecessarily, this means that some packet  $q$  of size  $\ell_j$  for some  $j$  is started in  $MAIN(s)$  at time  $\tau \leq t$  and running at  $t$ . As  $MAIN(s)$  does not complete any packet in  $(u, v]$ , the packet  $q$  is jammed by the fault at time  $v$ . This implies that  $j > i$ , as  $\ell_j > s(v - \tau) \geq v - t \geq \ell_i$ ; we also have  $t - \tau < \ell_j$ . Moreover,  $q$  is the only packet started by  $MAIN(s)$  in this block, thus it starts a phase.

As this phase is started by packet  $q$  of size  $\ell_j > \ell_i$ , the time  $\tau$  is  $i$ -good and  $C'_i \geq \tau$ . All packets  $ADV$  started before time  $C'_i + 4\ell_k/s$  are charged, as the packets in  $A$  are removed from  $ADV$  and packets in  $ADV$  are rescheduled only to later times. Packet  $p$  is started before  $v < \tau + \ell_j/s < C'_i + \ell_k/s$ , thus it is charged. It follows that  $p$  is charged by a forward charge. We now apply Lemma 4.2 again and observe that it implies that at  $\tau > t - \ell_j$  there are more than  $\ell_k/\ell_i$  packets of size  $\ell_i$  pending for  $MAIN(s)$ . This is in contradiction with the fact that at  $\tau$ ,  $MAIN(s)$  started a phase by  $q$  of size  $\ell_j > \ell_i$ .

**Main case 2: Non-empty block.** Otherwise,  $MAIN(s)$  completes a packet in the current block  $(u, v]$ .

Let  $Q$  be the set of packets completed by  $MAIN(s)$  in  $(u, v]$  that do not receive an up charge. Note that no packet in  $Q$  receives a forward charge, as the modified  $ADV$  contains no packets before  $u$ , so packets in  $Q$  either get a back charge or no charge at all. Let  $P$  be the set of packets completed in  $ADV$  in  $(u, v]$  that are not charged by an up charge. Note that  $P$  includes packets charged by a forward charge and uncharged packets, as no packets are charged to a previous block.

We first assign packets in  $P$  to packets in  $Q$  so that for each packet  $q \in Q$  the total size of packets assigned to  $q$  is at most  $\ell(q)$ . Formally, we iteratively define a provisional assignment  $f : P \rightarrow Q$  such that  $\ell(f^{-1}(q)) \leq \ell(q)$  for each  $q \in Q$ .

**Provisional assignment.** We maintain a set  $O \subseteq Q$  of *occupied* packets that we do not use for a future assignment. Whenever we assign a packet  $p$  to  $q \in Q$  and  $\ell(q) - \ell(f^{-1}(q)) < \ell(p)$ , we add  $q$  to  $O$ . This rule guarantees that each packet  $q \in O$  has  $\ell(f^{-1}(q)) > \ell(q)/2$ .

We process packets in  $P$  in the order decreasing sizes as follows. We take an arbitrary packet  $p \in P$  of the maximum size  $\ell(p)$  and choose an arbitrary packet  $q \in Q \setminus O$  such that  $\ell(q) \geq \ell(p)$ ; we prove in Lemma 4.4 below that such a  $q$  exists. We assign  $p$  to  $q$ , that is, we set  $f(p) = q$ . Furthermore, as described above, if  $\ell(q) - \ell(f^{-1}(q)) < \ell(p)$ , we add  $q$  to  $O$ . Continue until all packets are assigned.

If a packet  $p$  is assigned to  $q$  and  $q$  is not put in  $O$ , it follows that  $\ell(q) - \ell(f^{-1}(q)) \geq \ell(p)$ . This implies that after the next packet  $p'$  is assigned to  $q$ , we have  $\ell(q) \geq \ell(f^{-1}(q))$ , as the packets are processed from the largest one and thus  $\ell(p') \leq \ell(p)$ . It follows that at the end we obtain a valid provisional assignment.

**Lemma 4.4.** *The assignment process above assigns all packets in  $P$ .*

*Proof.* For each size  $\ell_j$  we show that all packets of size  $\ell_j$  in  $P$  are assigned, which is clearly sufficient. We fix the size  $\ell_j$  and define a few quantities.

Let  $n$  denote the number of packets of size  $\ell_j$  in  $P$ . Let  $o$  denote the total *occupied size*, defined as  $o = \ell(O) + \sum_{q \in Q \setminus O} \ell(f^{-1}(q))$  at the time just before we start assigning the packets of size  $\ell_j$ . Note that the rule for adding packets to  $O$  implies that  $\ell(f^{-1}(Q)) \geq o/2$ . Let  $a$  denote the current total *available size* defined as  $a = \sum_{q \in Q \setminus O: \ell(q) \geq \ell_j} (\ell(q) - \ell(f^{-1}(q)))$ .

First we claim that it is sufficient to show that  $a > (2n - 2)\ell_j$  before we start assigning the packets of size  $\ell_j$ . As long as  $a > 0$ , there is a packet  $q \in Q \setminus O$  of size at least  $\ell_j$  and thus we may assign the next packet (and, as noted before, actually  $a \geq \ell_j$ , as otherwise  $q \in O$ ). Furthermore, assigning a packet  $p$  of size  $\ell_j$  to  $q$  decreases  $a$  by  $\ell_j$  if  $q$  is not added to  $O$  and by less than  $2\ell_j$  if  $q$  is added to  $O$ . Altogether, after assigning the first  $n - 1$  packets,  $a$  decreases by less than  $(2n - 2)\ell_j$ , thus we still have  $a > 0$ , and we can assign the last packet. The claim follows.

We now split the analysis into two cases, in each we prove that the available space  $a$  is sufficiently large before assigning the packets of size  $\ell_j$ .

In the first case, we suppose that a packet of size  $\ell_j$  is pending for  $MAIN(s)$  all times in  $[u, v)$ . Let  $z$  be the total size of packets of size at least  $\ell_j$  charged by up charges in this block. The size of packets in  $P$  already assigned is at least  $\ell(f^{-1}(Q)) \geq o/2$  and we have  $n$  yet unassigned packets of size  $\ell_j$  in  $P$ . As  $ADV$  has to schedule all these packets and the packets with up charges in this block, its size satisfies  $v - u \geq \ell(P) + z \geq n\ell_j + o/2 + z$ . Now consider the schedule of  $MAIN(s)$  in this block. By Lemma 2.2, there is no end of phase in  $(u, v)$  and jobs smaller than  $\ell_j$  scheduled by  $MAIN(s)$  have total size less than  $2\ell_j$ . All the other completed packets contribute to one of  $a$ ,  $o$ , or  $z$ . Using Lemma 2.1, the previous bound on  $v - u$  and  $s \geq 4$ , the total size of completed packets is at least  $s(v - u)/2 \geq 2n\ell_j + o + 2z$ . Hence  $a > (2n\ell_j + o + 2z) - 2\ell_j - o - z \geq (2n - 2)\ell_j$ , which completes the proof of the lemma in this case.

Otherwise, in the second case, there is a time in  $[u, v)$  when no packet of size  $\ell_j$  is pending for  $MAIN(s)$ . Let  $\tau$  be the supremum of times  $\tau' \in [u, v]$  such that  $MAIN(s)$  has no pending packet of size at least  $\ell_j$  at time  $\tau'$ ; if no such  $\tau'$  exists we set  $\tau = u$ . Let  $t$  be the time when the adversary starts the first packet  $p$  of size  $\ell_j$  from  $P$ .

Since  $p$  is charged using a forward charge or  $p$  is not charged, we can apply Lemma 4.2 or 4.3, which implies that packets of size  $\ell_j$  are pending for  $MAIN(s)$  from time  $t - 3\ell_k$  till at least  $v$ . By the case condition, there is a time in  $[u, v)$  when no packet of size  $\ell_j$  is pending, and this time is thus before  $t - 3\ell_k$ , implying  $u < t - 3\ell_k$ . The definition of  $\tau$  now implies that  $\tau \leq t - 3\ell_k$ .

Towards bounding  $a$ , we show that (i)  $MAIN(s)$  runs a limited amount of small packets after  $\tau$  and thus  $a + o$  is large, and that (ii)  $f^{-1}(Q)$  contains only packets run by  $ADV$  from  $\tau$  on, and thus  $o$  is small.

We claim that the total size of packets smaller than  $\ell_j$  completed in  $MAIN(s)$  in  $(\tau, v]$  is less than  $3\ell_k$ . This claim is similar to Lemma 2.2 and we also argue similarly. Let  $\tau_1 < \tau_2 < \dots < \tau_\alpha$  be all the ends of phases in  $(\tau, v)$  (possibly there is none, then  $\alpha = 0$ ); also let  $\tau_0 = \tau$ . For  $i = 1, \dots, \alpha$ , let  $r_i$  denote the packet started by  $MAIN(s)$  at  $\tau_i$ ; note that  $r_i$  exists since after  $\tau$  there is a pending packet at any time in  $[\tau, v]$  by the definition of  $\tau$ . First note that any packet started at or after time  $\tau_\alpha + \ell_k/s$  has size at least  $\ell_j$ , as such a packet is pending and satisfies the condition in Step (3) of the algorithm. Thus the total amount of the small packets completed in  $(\tau_\alpha, v]$  is less than  $\ell_k + \ell_{k-1} < 2\ell_k$ . The claim now follows for  $\alpha = 0$ . Otherwise, as there is no fault in  $(u, v)$ , at  $\tau_i$ ,  $i = 1, \dots, \alpha$ , Step (4) of

the algorithm is reached and thus no packet of size at most  $s(\tau_i - \tau_{i-1})$  is pending. In particular, this implies that  $\ell(r_i) > s(\tau_i - \tau_{i-1})$  for  $i = 1, \dots, \alpha$ . This also implies that the amount of the small packets completed in  $(\tau_0, \tau_1]$  is less than  $\ell_k$  and the claim for  $\alpha = 1$  follows. For  $\alpha \geq 2$  first note that by Lemma 2.2(i),  $s(\tau_i - \tau_{i-1}) \geq \ell_j$  for all  $i = 2, \dots, \alpha$  and thus  $r_i$  is not a small packet. Thus for  $i = 3, \dots, \alpha$ , the amount of small packets in  $(\tau_{i-1}, \tau_i]$  is at most  $s(\tau_{i-1} - \tau_i) - \ell(r_{i-1}) < \ell(r_i) - \ell(r_{i-1})$ . The amount of small packets completed in  $(\tau_1, \tau_2]$  is at most  $s(\tau_2 - \tau_1) < \ell(r_2)$  and the amount of small packets completed in  $(\tau_\alpha, v]$  is at most  $2\ell_k - \ell(r_\alpha)$ . Summing this together, the amount of small packets completed in  $(\tau_1, v]$  is at most  $2\ell_k$  and the claim follows.

Let  $z$  be the total size of packets of size at least  $\ell_j$  charged by up charges in this block and completed by  $MAIN(s)$  after  $\tau$ . After  $\tau$ ,  $MAIN(s)$  processes packets of total size more than  $s(v - \tau) - \ell_k$  and all of these packets contribute to one of  $a$ ,  $o$ ,  $z$ , or the volume of less than  $3\ell_k$  of small packets from the claim above. Thus, using  $s \geq 4$ , we get

$$a > 4(v - \tau) - o - z - 4\ell_k. \quad (4.1)$$

Now we derive two lower bounds on  $v - \tau$  using  $ADV$  schedule.

Observe that no packet contributing to  $z$  except for possibly one (the one possibly started by  $MAIN(s)$  before  $\tau$ ) is started by  $ADV$  before  $\tau$ , as otherwise it would be pending for  $MAIN(s)$  just before  $\tau$ , contradicting the definition of  $\tau$ .

Also, observe that in  $(u, \tau]$ ,  $ADV$  runs no packet  $p \in P$  with  $\ell(p) > \ell_j$ : For a contradiction, assume that such a  $p$  exists. As  $\tau \leq C_{j'}$  for any  $j' > j$ , such a  $p$  is charged. As  $p \in P$ , it is charged by a forward charge. However, then Lemma 4.2 implies that at all times between the start of  $p$  in  $ADV$  and  $v$  a packet of size  $\ell(p)$  is pending for  $MAIN(s)$ ; in particular such a packet is pending in the interval before  $\tau$ , contradicting the definition of  $\tau$ .

These two observations imply that in  $[\tau, v]$ ,  $ADV$  starts and completes all the assigned packets from  $P$ , the  $n$  packets of size  $\ell_j$  from  $P$ , and all packets except possibly one contributing to  $z$ . This gives  $v - \tau \geq \ell(f^{-1}(Q)) + n\ell_j + z - \ell_k \geq o/2 + n\ell_j + z - \ell_k$ .

To obtain the second bound, we observe that the  $n$  packets of size  $\ell_j$  from  $P$  are scheduled in  $[t, v]$  and together with  $t \geq \tau + 3\ell_k$  we obtain  $v - \tau = v - t + t - \tau \geq n\ell_j + 3\ell_k$ .

Summing the two bounds on  $v - \tau$  and multiplying by two we get  $4(v - \tau) \geq 4n\ell_j + 4\ell_k + o + 2z$ . Summing with (4.1) we get  $a > 4n\ell_j + z \geq 4n\ell_j$ . This completes the proof of the second case.  $\square$

As a remark, note that in the previous proof, the first case deals with blocks after  $C_j$ , it is the typical and tight case. The second case deals mainly with the block containing  $C_j$ . (The blocks that are before  $C_j$  may fall in either of the cases.) This brings some technical difficulties, but it is a case which has slack. This is similar to the situation in the local analysis using Master Theorem.

**Modifying the adversary schedule.** Now all the packets from  $P$  are provisionally assigned by  $f$  and for each  $q \in Q$  we have that  $\ell(f^{-1}(q)) \leq \ell(q)$ .

We process all packets  $q$  completed by  $MAIN(s)$  in  $(u, v]$  one by one according to one of the following three cases; in each case we remove from  $ADV$  one or more packets with total size at most  $\ell(q)$ .

If  $q \notin Q$  then the definition of  $P$  and  $Q$  implies that  $q$  is charged by an up charge from some packet  $p \notin P$  of the same size. We remove  $p$  from  $ADV$ .

If  $q \in Q$  does not receive a charge, we remove  $f^{-1}(q)$  from  $ADV$ . We have  $\ell(f^{-1}(q)) \leq \ell(q)$ , so the size is as required. If any packet  $p \in f^{-1}(q)$  is charged (necessarily by a forward charge), we remove this charge.

If  $q \in Q$  receives a charge, it is a back charge from some packet  $p$  of the same size. We remove  $p$  from  $ADV$  and in the interval where  $p$  was scheduled, we schedule packets from  $f^{-1}(q)$  in an arbitrary order. As  $\ell(f^{-1}(q)) \leq \ell(q)$ , this is feasible. If any packet  $p \in f^{-1}(q)$  is charged, we keep its charge to the same packet in  $MAIN(s)$ ; the charge was necessarily a forward charge, so it leads to some later block.

After we have processed all the packets  $q$ , we have modified  $ADV$  by removing an allowed total size of packets and rescheduling the remaining packets in  $(u, v]$  so that any remaining charges go to later blocks. This completes processing of the block  $(u, v]$  and thus also the proof of 1-competitiveness.

## 5 Lower Bounds

In this section we study lower bounds on the speed necessary to achieve 1-competitiveness. Note that previously only lower bounds on competitiveness with speed 1 were known.

We start with a lower bound of 2 which holds even for the divisible case. It follows that our algorithm  $DIV$  and the algorithm in Jurdzinski *et al.* [7] are optimal.



**Theorem 5.1.** *There is no 1-competitive deterministic online algorithm running with speed  $s < 2$ , even if packets have sizes only 1 and  $\ell$  for  $\ell > 2s/(2-s)$  and all of them are released at time 0.*

*Proof.* For a contradiction, consider an algorithm  $ALG$  running with speed  $s < 2$  that is claimed to be 1-competitive with additive constant  $A$  where  $A$  may depend on  $\ell$ . At time 0 the adversary releases  $N_1 = \lceil A/\ell \rceil + 1$  packets of size  $\ell$  and  $N_0 = \lceil (2/s) \cdot N_1 \cdot (s-1) \cdot \ell^2 + (2/s) \cdot \ell \cdot (A+1) \rceil$  packets of size 1. These are all packets in the instance.

The adversary's strategy works by blocks where a block is a time interval between two faults, where the first block begins at time 0. The adversary ensures that in each such block  $ALG$  completes no packet of size  $\ell$  and moreover  $ADV$  either completes an  $\ell$ -sized packet or completes more 1's (packets of size 1) than  $ALG$ .

Let  $t$  be the time of the last fault; initially  $t = 0$ . Let  $\tau \geq t$  be the time when  $ALG$  starts the first  $\ell$ -sized packet after  $t$ ; we set  $\tau = \infty$  if it does not happen. Note that we use here that  $ALG$  is deterministic. In a block beginning at time  $t$ , the adversary proceeds according to the first case below that applies.

- (D1) If  $ADV$  has less than  $2\ell/s$  pending packets of size 1, then the end of the schedule is at  $t$ .
- (D2) If  $ADV$  has all packets of size  $\ell$  completed, then it stops the current process and issues a fault at times  $t+1, t+2, \dots$ . Between each two consecutive faults after  $t$  it completes one packet of size 1 and it continues issuing faults until it has no pending packet of size 1. Then there is the end of the schedule. Clearly,  $ALG$  may complete only packets of size 1 after  $t$  as  $\ell > 2s/(2-s) > s$ .
- (D3) If  $\tau \geq t + \ell/s - 2$ , then the next fault is at time  $t + \ell$ . In the current block, the adversary completes a packet  $\ell$ .  $ALG$  completes at most  $s \cdot \ell$  packets of size 1 and then it possibly starts  $\ell$  at  $\tau$  (if  $\tau < t + \ell$ ) which is jammed, since it would be completed at

$$\tau + \frac{\ell}{s} \geq t + \frac{2\ell}{s} - 2 = t + \ell + \left(\frac{2}{s} - 1\right)\ell - 2 > t + \ell$$

where the last inequality follows from  $\left(\frac{2}{s} - 1\right)\ell > 2$  which is equivalent to  $\ell > 2s/(2-s)$ . Thus the  $\ell$ -sized packet would be completed after the fault.

- (D4) Otherwise, if  $\tau < t + \ell/s - 2$ , then the next fault is at time  $\tau + \ell/s - \varepsilon$  for a small enough  $\varepsilon > 0$ . In the current block,  $ADV$  completes as many packets of size 1 as it can, that is  $\lfloor \tau + \ell/s - \varepsilon - t \rfloor$  packets of size 1; note that by Case (D1),  $ADV$  has enough 1's pending. Again, the algorithm does not complete the packet of size  $\ell$  started at  $\tau$ , because it would be finished at  $\tau + \ell/s$ .

First notice that the process above ends, since in each block the adversary completes a packet. We now show  $L_{ADV} > L_{ALG} + A$  which contradicts the claimed 1-competitiveness of  $ALG$ .

If the adversary's strategy ends in Case (D2), then  $ADV$  has all  $\ell$ 's completed and then it schedules all 1's, thus  $L_{ADV} = N_1 \cdot \ell + N_0 > A + N_0$ . However,  $ALG$  does not complete any  $\ell$ -sized packet and hence  $L_{ALG} \leq N_0$  which concludes this case.

Otherwise, the adversary's strategy ends in Case (D1). We first claim that in a block  $(t, t']$  created in Case (D4),  $ADV$  finishes more 1's than  $ALG$ . Indeed, let  $o$  be the number of 1's completed by  $ALG$  in  $(t, t']$ . Then  $\tau \geq t + o/s$  where  $\tau$  is from the adversary's strategy in  $(t, t']$ , and we also have  $o < \ell - 2s$  or equivalently  $\ell > o + 2s$ , because  $\tau < t + \ell/s - 2$  in Case (D4). The number of 1's scheduled by  $ADV$  is

$$\begin{aligned} \left\lfloor \tau + \frac{\ell}{s} - \varepsilon - t \right\rfloor &\geq \left\lfloor t + \frac{o}{s} + \frac{\ell}{s} - \varepsilon - t \right\rfloor \geq \left\lfloor \frac{o}{s} + \frac{o + 2s}{s} - \varepsilon \right\rfloor = \left\lfloor \frac{2}{s}o + 2 - \varepsilon \right\rfloor \\ &\geq \left\lfloor \frac{2}{s}o + 1 \right\rfloor \geq o + 1 \end{aligned}$$

and we proved the claim.

Let  $\alpha$  be the number of blocks created in Case (D3); note that  $\alpha \leq N_1$ , since in each such block  $ADV$  finishes one  $\ell$ -sized packet.  $ALG$  completes at most  $s\ell$  packets of size 1 in such a block, thus  $L_{ADV}((u, v]) - L_{ALG}((u, v]) \geq (1-s) \cdot \ell$  for a block  $(u, v]$  created in Case (D3).

Let  $\beta$  be the number of blocks created in Case (D4). We have

$$\beta > \frac{s}{2\ell} \cdot \left(N_0 - \frac{2\ell}{s}\right) = \frac{s \cdot N_0}{2\ell} - 1,$$

because in each such block  $ADV$  schedules less than  $2\ell/s$  packets of size 1 and less than  $2\ell/s$  of these packets are pending at the end. By the claim above, we have  $L_{ADV}((u, v]) - L_{ALG}((u, v]) \geq 1$  for a block  $(u, v]$  created in Case (D4).

Summing over all blocks and using the value of  $N_0$  we get

$$\begin{aligned} L_{ADV} - L_{ALG} &\geq \alpha \cdot (1 - s) \cdot \ell + \beta \\ &> N_1 \cdot (1 - s) \cdot \ell + \frac{s \cdot N_0}{2\ell} - 1 \\ &\geq N_1 \cdot (1 - s) \cdot \ell + N_1 \cdot (s - 1) \cdot \ell + A + 1 - 1 = A \end{aligned}$$

where we used  $s \geq 1$  which we may suppose w.l.o.g. This concludes the proof.  $\square$

Our main lower bound of  $\phi + 1 = \phi^2 \approx 2.618$  generalizes the construction of Theorem 5.1 for more packet sizes, which are no longer divisible. Still, we make no use of release times.

**Theorem 5.2.** *There is no 1-competitive deterministic online algorithm running with speed  $s < \phi + 1$ , even if all packets are released at time 0.*

*Proof.* The adversary chooses  $\varepsilon > 0$  small enough and  $k \in \mathbb{N}$  large enough so that  $s < \phi + 1 - 1/\phi^{k-1}$ . For convenience, the smallest size in the instance is  $\varepsilon$  instead of 1. There will be  $k + 1$  packet sizes in the instance, namely  $\ell_0 = \varepsilon$ , and  $\ell_i = \phi^{i-1}$  for  $i = 1, \dots, k$ .

Suppose for a contradiction that there is an algorithm  $ALG$  running with speed  $s < \phi + 1$  that is claimed to be 1-competitive with additive constant  $A$  where  $A$  may depend on  $\ell_i$ 's, in particular on  $\varepsilon$  and  $k$ . The adversary issues  $N_i$  packets of size  $\ell_i$  at time 0, for  $i = 0, \dots, k$ ;  $N_i$ 's are chosen so that  $N_0 \gg N_1 \gg \dots \gg N_k$ . These are all packets in the instance.

More precisely,  $N_i$ 's are defined inductively so that it holds  $N_k > A/\ell_k$ ,  $N_i > \ell_k \phi^s \sum_{j=i+1}^k N_j + A/\ell_i$  for  $i = k - 1, \dots, 1$ , and finally  $N_0 > \phi \ell_k (A + 1 + s \phi \ell_k \sum_{i=1}^k N_i) / \varepsilon^2$ .

The adversary's strategy works by blocks where a block is again a time interval between two faults and the first block begins at time 0. Let  $t$  be the time of the last fault; initially  $t = 0$ . Let  $\tau_i \geq t$  be the time when  $ALG$  starts the first packet of size  $\ell_i$  after  $t$ ; we set  $\tau_i = \infty$  if it does not happen. Note that we use here that  $ALG$  is deterministic. Let  $\tau_{\geq i} = \min_{j \geq i} \tau_j$  be the time when  $ALG$  starts the first packet of size at least  $\ell_i$  after  $t$ . Let  $P_{ADV}(i)$  be the total size of  $\ell_i$ 's (packets of size  $\ell_i$ ) pending for the adversary at time  $t$ .

In a block beginning at time  $t$ , the adversary proceeds according to the first case below that applies. Each case has an intuitive explanation which we make precise later.

(B1) If  $P_{ADV}(0) < \phi \ell_k$  (i.e., there are less than  $\phi \ell_k / \varepsilon$  packets of size  $\varepsilon$  pending for  $ADV$ ), then the end of the schedule is at time  $t$ .

Lemma 5.3 below shows that in blocks in which  $ADV$  schedules  $\varepsilon$ 's it completes more than  $ALG$  in terms of total size. It follows that the schedule of  $ADV$  has much bigger total size for  $N_0$  large enough, since the adversary scheduled nearly all packets of size  $\varepsilon$ ; see Lemma 5.8.

(B2) If there is  $i \geq 1$  such that  $P_{ADV}(i) = 0$ , then  $ADV$  stops the current process and continues by Strategy FINISH described below.

(B3) If  $\tau_1 < t + \ell_1 / (\phi \cdot s)$ , then the next fault occurs at time  $\tau_1 + \ell_1 / s - \varepsilon$ , so that  $ALG$  does not finish the first  $\ell_1$ -sized packet.  $ADV$  schedules as many  $\varepsilon$ 's as it can.

In this case,  $ALG$  schedules  $\ell_1$  too early and in Lemma 5.3 we show that the total size of packets completed by  $ADV$  is larger than the total size of packets completed by  $ALG$ .

(B4) If  $\tau_{\geq 2} < t + \ell_2 / (\phi \cdot s)$ , then the next fault is at time  $\tau_{\geq 2} + \ell_2 / s - \varepsilon$ , so that  $ALG$  does not finish the first packet of size at least  $\ell_2$ .  $ADV$  again schedules as many  $\varepsilon$ 's as it can. (We remark that  $\tau_{\geq 2} < \tau_1$  in this case.) Similarly as in the previous case,  $ALG$  starts  $\ell_2$  or a larger packet too early and we show that  $ADV$  completes more in terms of size than  $ALG$ , again using Lemma 5.3.

(B5) If there is  $1 \leq i < k$  such that  $\tau_{\geq i+1} < \tau_i$ , then we choose the smallest such  $i$  and the next fault is at time  $t + \ell_i$ .  $ADV$  schedules a packet of size  $\ell_i$ .

Intuitively, this case means that  $ALG$  skips  $\ell_i$  and schedules  $\ell_{i+1}$  (or a larger packet) earlier. Lemma 5.5 shows that the algorithm cannot finish its first packet of size at least  $\ell_{i+1}$  (thus it also does not schedule  $\ell_i$ ) provided that this case is not triggered for a smaller  $i$ , or previous cases are not triggered.

(B6) Otherwise, the next fault occurs at  $t + \ell_k$  and  $ADV$  schedules a packet of size  $\ell_k$  in this block. Lemma 5.6 shows that  $ALG$  cannot complete an  $\ell_k$ -sized packet in this block.

We remark that the process above eventually ends either in Case (B1), or in Case (B2), since in each block  $ADV$  schedules a packet. Also note that the length of each block is at most  $\phi \ell_k$ .

We describe Strategy FINISH, started in Case (B2). Let  $i$  be the smallest index  $i' \geq 1$  such that  $P_{ADV}(i') = 0$ . For brevity, we call a packet of size at least  $\ell_i$  *long*, and a packet of size  $\ell_j$  with

$1 \leq j < i$  short; note that  $\varepsilon$ 's are not short packets. In a nutshell,  $ADV$  tries to schedule all short packets, while preventing the algorithm from completing any long packet. Similarly to Cases (B3) and (B4), if  $ALG$  is starting a long packet too early,  $ADV$  schedules  $\varepsilon$ 's and gains in terms of total size.

Adversary's Strategy FINISH works again by blocks. Let  $t$  be the time of the last fault. Let  $\tau \geq t$  be the time when  $ALG$  starts the first long packet after  $t$ ; we set  $\tau = \infty$  if it does not happen. The adversary proceeds according to the first case below that applies:

- (F1) If  $P_{ADV}(0) < \phi \ell_k$ , then the end of the schedule is at time  $t$ .
- (F2) If  $ADV$  has no pending short packet, then the strategy FINISH ends and  $ADV$  issues a fault at times  $t + \varepsilon, t + 2\varepsilon, \dots$ . Between each two consecutive faults after  $t$  it completes one packet of size  $\varepsilon$  and it continues issuing faults until it has no pending  $\varepsilon$ . Then there is the end of the schedule. Clearly,  $ALG$  may complete only  $\varepsilon$ 's after  $t$  if  $\varepsilon$  is small enough. Note that for  $i = 1$  this case is immediately triggered, as  $\ell_0$ -sized packets are not short, hence there are no short packets whatsoever.
- (F3) If  $\tau < t + \ell_i/(\phi \cdot s)$ , then the next fault is at time  $\tau + \ell_i/s - \varepsilon$ , so that  $ALG$  does not finish the first long packet.  $ADV$  schedules as many  $\varepsilon$ 's as it can. Note that the length of this block is less than  $\ell_i/(\phi \cdot s) + \ell_i/s \leq \phi \ell_k$ . Again, we show that  $ADV$  completes more in terms of size using Lemma 5.3.
- (F4) Otherwise,  $\tau \geq t + \ell_i/(\phi \cdot s)$ .  $ADV$  issues the next fault at time  $t + \ell_{i-1}$ . Let  $j$  be the largest  $j' < i$  such that  $P_{ADV}(j') > 0$ .  $ADV$  schedules a packet of size  $\ell_j$  which is completed as  $j \leq i - 1$ . Lemma 5.7 shows that  $ALG$  does not complete the long packet started at  $\tau$ .

Again, in each block  $ADV$  completes a packet, thus Strategy FINISH eventually ends. Note that the length of each block is less than  $\phi \ell_k$ .

We now prove the lemmata mentioned above. In the following,  $t$  is the beginning of the considered block and  $t'$  is the end of the block, i.e., the next fault after  $t$ . Recall that  $L_{ALG}((t, t'])$  is the total size of packets completed by  $ALG$  in  $(t, t']$ . We start with a general lemma that covers all cases in which  $ADV$  schedules many  $\varepsilon$ 's.

**Lemma 5.3.** *In Cases (B3), (B4), and (F3) it holds  $L_{ADV}((t, t']) \geq L_{ALG}((t, t']) + \varepsilon$ .*

*Proof.* Let  $i$  and  $\tau$  be as in Case (F3); we set  $i = 1$  and  $\tau = \tau_1$  in Case (B3), and  $i = 2$  and  $\tau = \tau_{\geq 2}$  in Case (B4). Note that the first packet of size (at least)  $\ell_i$  is started at  $\tau$  with  $\tau < t + \ell_i/(\phi \cdot s)$  and that the next fault occurs at time  $\tau + \ell_i/s - \varepsilon$ . Furthermore,  $P_{ADV}(0, t) \geq \phi \ell_k$  by Cases (B1) and (F1). As  $t' - t \leq \phi \ell_k$  it follows that  $ADV$  has enough  $\varepsilon$ 's to fill nearly the whole block with them, so in particular  $L_{ADV}((t, t']) > t' - t - \varepsilon$ .

Let  $a = L_{ALG}((t, t'])$ . Since  $ALG$  does not complete the  $\ell_i$ -sized packet we have  $\tau \geq t + a/s$  and thus also  $a < \ell_i/\phi$  as  $\tau < t + \ell_i/(\phi \cdot s)$ .

If  $a < \ell_i/\phi - 3s\varepsilon/\phi$  which is equivalent to  $\ell_i > \phi \cdot a + 3s\varepsilon$ , then we show the required inequality by the following calculation:

$$L_{ADV}((t, t']) + \varepsilon > t' - t = \tau + \frac{\ell_i}{s} - \varepsilon - t \geq \frac{a}{s} + \frac{\ell_i}{s} - \varepsilon > \frac{a + \phi \cdot a + 3s\varepsilon}{s} - \varepsilon > a + 2\varepsilon.$$

Otherwise,  $a$  is nearly  $\ell_i/\phi$  and thus large enough. Then we get

$$L_{ADV}((t, t']) + \varepsilon > t' - t = \tau + \frac{\ell_i}{s} - \varepsilon - t \geq \frac{a}{s} + \frac{\ell_i}{s} - \varepsilon > \frac{a}{s} + \frac{\phi a}{s} - \varepsilon > a + 2\varepsilon$$

where the penultimate inequality follows by  $\ell_i > \phi a$ , and the last inequality holds as  $(1 + \phi)a/s > a + 3\varepsilon$  for  $\varepsilon$  small enough and  $a \geq \ell_i/\phi - 3s\varepsilon/\phi$ .  $\square$

For brevity, we inductively define  $S_0 = \phi - 1$  and  $S_i = S_{i-1} + \ell_i$  for  $i = 1, \dots, k$ . Thus  $S_i = \sum_{j=1}^i \ell_j + \phi - 1$  and a calculation shows  $S_i = \phi^{i+1} - 1$ . We prove a useful observation.

**Lemma 5.4.** *Fix  $j \geq 2$ . If Case (B3) and Case (B5) for  $i < j$  are not triggered in the block, then  $\tau_{i+1} \geq t + S_i/s$  for each  $i < j$ .*

*Proof.* We have  $\tau_1 \geq t + \ell_1/(\phi \cdot s) = t + (\phi - 1)/s$  by Case (B3) and  $\tau_{i+1} \geq \tau_i + \ell_i/s$  for any  $i < j$ , since Case (B5) was not triggered for  $i < j$  and the first  $\ell_i$ -sized packet needs to be finished before starting the next packet. Summing the bounds gives the inequalities in the lemma.  $\square$

**Lemma 5.5.** *In Case (B5) the algorithm does not complete any packet of size  $\ell_i$  or larger.*

*Proof.* Recall that we have  $\tau_{\geq i+1} < \tau_i$ , thus the first packet  $p$  of size at least  $\ell_i$  has size at least  $\ell_{i+1}$ . It suffices to prove

$$\tau_{\geq i+1} + \frac{\ell_{i+1}}{s} - t > \ell_i , \quad (5.1)$$

which means that  $p$  would be completed after the next fault at time  $t + \ell_i$ .

We start with the case  $i = 1$  in which  $\tau_{\geq 2} < \tau_1$ . Since Case (B4) was not triggered, we have  $\tau_{\geq 2} \geq t + \ell_2/(\phi \cdot s) = t + 1/s$ . We show (5.1) by the following calculation:

$$\tau_{\geq 2} + \frac{\ell_2}{s} - t \geq \frac{1}{s} + \frac{\ell_2}{s} = \frac{1 + \phi}{s} > 1 = \ell_1 ,$$

where the strict inequality holds by  $s < \phi + 1$ .

Now consider the case  $i \geq 2$ . By the minimality of  $i$  satisfying the condition of Case (B5) we use Lemma 5.4 for  $j = i$  to get  $\tau_{i-1} \geq t + S_{i-2}/s$ . Since a packet  $\ell_{i-1}$  is started at  $\tau_{i-1}$ , it holds  $\tau_{\geq i+1} \geq t + (S_{i-2} + \ell_{i-1})/s = t + S_{i-1}/s$ . Thus

$$\tau_{\geq i+1} + \frac{\ell_{i+1}}{s} - t \geq \frac{S_{i-1} + \ell_{i+1}}{s} = \frac{\phi^i - 1 + \phi^i}{s} = \frac{\phi^{i+1} + \phi^{i-2} - 1}{s} \geq \frac{\phi^{i+1}}{s} \geq \phi^{i-1} = \ell_i ,$$

where the penultimate inequality holds by  $i \geq 2$  and the last inequality by  $s < \phi + 1$ . (We remark that the penultimate inequality has a significant slack for  $i > 2$ .)  $\square$

**Lemma 5.6.** *In Case (B6) ALG does not complete a packet of size  $\ell_k$ .*

*Proof.* It suffices to prove

$$\tau_k > t + \left(1 - \frac{1}{s}\right) \ell_k , \quad (5.2)$$

since then ALG completes the first  $\ell_k$ -sized packet at

$$\tau_k + \frac{\ell_k}{s} > t + \left(1 - \frac{1}{s}\right) \ell_k + \frac{\ell_k}{s} = t + \ell_k ,$$

thus after the fault.

Since Case (B3) and Case (B5) are not triggered, we use Lemma 5.4 for  $j = k$  to get  $\tau_k \geq t + S_{k-1}/s = t + (\phi^k - 1)/s$ . Recall that  $k$  is large enough so that  $s < \phi + 1 - 1/\phi^{k-1}$  which is equivalent to  $(\phi^k - 1)/s > (1 - 1/s)\phi^{k-1}$ . This shows (5.2), because  $\tau_k - t \geq (\phi^k - 1)/s$  and  $\ell_k = \phi^{k-1}$ .  $\square$

**Lemma 5.7.** *In Case (F4) ALG does not complete any long packet.*

*Proof.* Recall that the first long packet  $p$  is started at  $\tau$  and it has size of at least  $\ell_i$ , thus it would be completed at least at  $\tau + \ell_i/s$ . We show  $\tau + \ell_i/s - t > \ell_{i-1}$  by the following calculation:

$$\tau + \frac{\ell_i}{s} - t \geq \frac{\ell_i}{\phi \cdot s} + \frac{\ell_i}{s} = \frac{\phi \ell_i}{s} > \frac{\ell_i}{\phi} = \ell_{i-1} ,$$

where the strict inequality holds by  $s < \phi + 1$ . This implies that the long packet  $p$  would be completed after the next fault at time  $t + \ell_{i-1}$ .  $\square$

We are ready to prove that at the end of the schedule  $L_{ADV} > L_{ALG} + A$  holds, which contradicts the claimed 1-competitiveness of ALG. We inspect all the cases in which the instances may end, starting with Cases (B1) and (F1). We remark that we use only crude bounds to keep the analysis simple.

**Lemma 5.8.** *If the schedule ends in Case (B1) or (F1), we have  $L_{ADV} > L_{ALG} + A$ .*

*Proof.* Recall that each block  $(t, t']$  has length of at most  $\phi \ell_k$ , thus  $L_{ALG}((t, t']) \leq s \phi \ell_k$  and  $L_{ADV}((t, t']) \leq \phi \ell_k$ .

We call a block in which ADV schedules many  $\varepsilon$ 's *small*, other blocks are *big*. Note that Cases (B3), (B4), and (F3) concern small blocks, whereas Cases (B5), (B6), and (F4) concern big blocks.

Using Lemma 5.3, we have that in each small block  $(t, t']$  it holds  $L_{ADV}((t, t']) \geq L_{ALG}((t, t']) + \varepsilon$ . Let  $\beta$  be the number of small blocks. We observe that

$$\beta \geq \frac{\left(N_0 - \frac{\phi \ell_k}{\varepsilon}\right) \varepsilon}{\phi \ell_k} ,$$

because in each such block ADV schedules at most  $\phi \ell_k / \varepsilon$  packets of size  $\varepsilon$  and  $P_{ADV}(0) < \phi \ell_k$  at the end in Cases (B1) and (F1).

The number of big blocks is at most  $\sum_{i=1}^k N_i$ , since in each such block  $ADV$  schedules a packet of size at least  $\ell_1$ . For each such block we have  $L_{ADV}((t, t']) - L_{ALG}((t, t']) \geq -s\phi\ell_k$  which is only a crude bound, but it suffices for  $N_0$  large enough.

Summing over all blocks we obtain

$$\begin{aligned} L_{ADV} - L_{ALG} &\geq \beta\varepsilon - s\phi\ell_k \sum_{i=1}^k N_i \geq \frac{\left(N_0 - \frac{\phi\ell_k}{\varepsilon}\right)\varepsilon^2}{\phi\ell_k} - s\phi\ell_k \sum_{i=1}^k N_i \\ &> A + s\phi\ell_k \sum_{i=1}^k N_i - s\phi\ell_k \sum_{i=1}^k N_i = A, \end{aligned} \quad (5.3)$$

where (5.3) follows from  $N_0 > \phi\ell_k(A + 1 + s\phi\ell_k \sum_{i=1}^k N_i)/\varepsilon^2$ .  $\square$

It remains to prove the same for termination by Case (F2).

**Lemma 5.9.** *If Strategy FINISH ends in Case (F2), then  $L_{ADV} > L_{ALG} + A$ .*

*Proof.* Note that  $ADV$  schedules all short packets and all  $\varepsilon$ 's, i.e., those of size less than  $\ell_i$ . In particular, we have  $L_{ADV}(< i) \geq L_{ALG}(< i)$ .

Call a block in which  $ALG$  completes a packet of size at least  $\ell_i$  *bad*. As the length of any block is at most  $\phi\ell_k$  we get that  $L_{ALG}(\geq i, (t, t']) \leq s\phi\ell_k$  for a bad block  $(t, t']$ . Bad blocks are created only in Cases (B5) and (B6), but in each bad block  $ADV$  finishes a packet strictly larger than  $\ell_i$ ; note that here we use Lemmata 5.5 and 5.6. Hence the number of bad blocks is bounded by  $\sum_{j=i+1}^k N_j$ . As  $ADV$  completes all  $\ell_i$ 's we obtain

$$L_{ADV}(\geq i) - L_{ALG}(\geq i) \geq \ell_i N_i - \ell_k \phi s \sum_{j=i+1}^k N_j > \ell_i \ell_k \phi s \sum_{j=i+1}^k N_j + A - \ell_k \phi s \sum_{j=i+1}^k N_j \geq A,$$

where in the strict inequality we used  $N_k > A/\ell_k$  for  $i = k$  and  $N_i > \ell_k \phi s \sum_{j=i+1}^k N_j + A/\ell_i$  for  $i < k$ . By summing it with  $L_{ADV}(< i) \geq L_{ALG}(< i)$  we conclude that  $L_{ADV} > L_{ALG} + A$ .  $\square$

Using Lemmata 5.8 and 5.9 we conclude the proof of the theorem.  $\square$

## 6 Examples for $MAIN$ and $DIV$

In this section we show some hard instances for both algorithms, obtaining some lower bounds that match our upper bounds from previous sections.

### 6.1 Example with general packet sizes

We prove that for  $s \in [2, 4)$ , the ratio of  $MAIN(s)$  is at least  $4/s > 1$ , which shows that the speedup in Theorem 4.1 cannot be improved.

Choose a large enough integer  $y$ . There will be four packet sizes:  $1, x, y$  and  $z$  such that  $1 < x < y < z$ ,  $z = x + y - 1$ , and  $x = y \cdot (s - 2)/2 + 2$ ; as  $s \geq 2$  it holds  $x > 1$  and as  $s < 4$  we have  $x \leq y - 1$  for a large enough  $y$ .

We will have  $N$  phases again. At time 0 the adversary  $\varepsilon$  releases all  $N(y - 1)$  packets of size 1, all  $N$  packets of size  $y$  and a single packet of size  $z$  (never completed by either  $OPT$  or  $MAIN(s)$ ), whereas the packets of size  $x$  are released one per phase.

In each phase  $MAIN(s)$  completes, in this order:  $y - 1$  packets of size 1 and then a packet of size  $x$ , which has arrived just after the  $y - 1$  packets of size 1 are completed. Next, it will start a packet of size  $z$  and fail due to a jam. We show that  $OPT$  will complete a packet of size  $y$ . To this end, it is required that  $y < 2(x + y - 1)/s$ , or equivalently  $x > y \cdot (s - 2)/2 + 1$  which holds by the choice of  $x$ .

After these  $N$  phases, we will have jams every 1 unit of time, so that  $OPT$  can complete all the  $N(y - 1)$  packets of size 1, while  $MAIN(s)$  will be unable to complete any packet (of size  $y$  or larger). The ratio per phase is

$$\frac{OPT}{MAIN(s)} = \frac{y - 1 + y}{y - 1 + x} = \frac{2y - 1}{y - 1 + \frac{y \cdot (s - 2)}{2} + 2} = \frac{2y - 1}{\frac{y \cdot s}{2} + 1}$$

which tends to  $4/s$  as  $y \rightarrow \infty$ .

## 6.2 Example with two packet sizes

Our next example shows that both algorithms have ratio no smaller than 2 when  $s < 2$ , even if packet sizes are only 1 and  $\ell \geq \max\left\{s + \epsilon, \frac{\epsilon}{2-s}\right\}$  for an arbitrarily small  $\epsilon > 0$ . This matches upper bounds given in Theorem 3.4 for *MAIN*(2) and in Theorem 3.9 for *DIV*. We denote either algorithm by *ALG*.

There will be  $N$  phases, that all look the same: In each phase, issue one packet of size  $\ell$  and  $\ell$  packets of size 1, and have the phase end by a fault at time  $(2\ell - \epsilon)/s \geq \ell$  which holds by the bounds on  $\ell$ . Then *ALG* will complete all  $\ell$  packets of size 1 but will not complete the one of size  $\ell$ . By the previous inequality, *OPT* can complete the packet of size  $\ell$  within the phase. Once all  $N$  phases are over, the jams occur every 1 unit of time, which allows *OPT* completing all  $N\ell$  remaining packets of size 1. However, *ALG* is unable to complete any of the packets of size  $\ell$ . Thus the ratio is 2.

## 6.3 Example for *MAIN* in the divisible case

We give an example that shows that *MAIN* is not very good for divisible instances, in particular it is not 1-competitive for any speed  $s < 2.5$  and thus the bound in Theorem 3.5 is tight. In fact, we show that it is not better than  $4/3$ -competitive. We use packets of sizes 1,  $\ell$ , and  $2\ell$  and we take  $\ell$  sufficiently large compared to the given speed or competitive ratio. There are many packets of size 1 and  $2\ell$  available at the beginning, the packets of size  $\ell$  arrive at specific times where *MAIN* schedules them immediately.

The faults occur at times divisible by  $2\ell$ , so the optimum schedules one packet of size  $2\ell$  in each phase between two faults. We have  $N$  such phases,  $N(2\ell - 1)$  packets of size 1 and  $N$  packets of size  $2\ell$  available at the beginning. In each phase, *MAIN*( $s$ ) schedules  $2\ell - 1$  packets of size 1, then a packet of size  $\ell$  arrives and is scheduled, and then a packet of size  $2\ell$  is scheduled. The algorithm would need speed  $2.5 - 1/(2\ell)$  to complete it. So, for  $\ell$  large, the algorithm completes only packets of total size  $3\ell - 1$  per phase. After these  $N$  phases, we have faults after time 1, so the optimum schedules all packets of size 1. The optimum thus finishes all packets  $2\ell$  plus all small packets, a total of  $4\ell - 1$  per phase. Thus the ratio tends to  $4/3$  as  $\ell \rightarrow \infty$ .

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