NDMI025 – Randomized Algorithms (Pravděpodobnostní algoritmy) LS 2015 – Jiří Sgall

Homework 3 – March 30

Due date: April 13 before the lecture

Recitation session for this homework set will take place on April 16

Sign all the sheets either by your name or by your nickname.

Every exercise is worth 2 points, the starred one is a bonus exercise.

(1) A cat and a mouse each independently take a random walk on a general graph G similarly to the previous homework. (In each step, they both take a random step, i.e., both random walks are synchronized. The game ends if they land at the same vertex at the same time. If they traverse the same edge in the opposite directions, nothing happens.)

Find an upper bound (as small as you can) on the expected number of steps of the game depending on the number of vertices and edges of G, and perhaps on other properties of G if needed.

You may try to find a graph G and initial positions such that the expected length of the game is as large as possible. Obtaining asymptotically tight bounds is harder, leaving a gap of n^2 is good enough.

(2) Let G be a bipartite (multi)graph with $2n$ vertices created as a union of d independent uniformly random perfect matchings on the two partitions with n vertices each. Prove that for some $c > 0$ and d, with high probability G is an expander graph in the following sense: Every set S of vertices in one partition such that $|S| \leq n/2$ has at least $(1 + c)|S|$ neighbors (in the other partition).

Hint: Calculate the probability that each S of size k expands; a fairly rough bound is sufficient, although you need to be somewhat careful when $k = o(n)$.

(3) Let us work in the field $GF(2)$. Let A be a square non-zero $n \times n$ matrix. Let $x, y \in \{0,1\}^n$ be independent uniformly random vectors. Prove that $Pr_{x,y}[x^T Ay \neq 0] \geq 1/4$.

Is the same true over the real numbers for $x, y \in \{0,1\}^n$?

(4) Let us work in the field $GF(2)$. Suppose that the function $G : \{0,1\}^n \to \{0,1\}$ satisfies that $Pr_{x,y}[G(x+y) = G(x)+G(y)] \geq 1-\delta]$ for some small $\delta > 0$ (such as $\delta = 1/10$), where $x, y \in \{0,1\}^n$ are independent uniformly random vectors. Define $H(z)$ as the more frequent result of $G(z + x) + G(x)$ over $x \in \{0,1\}^n$. Prove that the function H is linear, i.e., $H(x + y) = H(x) + H(y)$ for all x and y.

Hint: Expand the expression $H(z)+H(t)$ for a fixed z and t according to the definition of H and prove that it is equal to $H(z + t)$ with positive probability. Recall that addition and subtracting denote the same operation over $GF(2)$.

(4*) For a sufficiently small δ show that $Pr_x[G(x) = H(x)] \geq 1 - 3\delta$ and H is the only linear function with this property.