NDMI025 – Randomized Algorithms (Pravděpodobnostní algoritmy) LS 2015 – Jiří Sgall

Homework 1 – Feb 23

Due date: March 9 before the lecture

Recitation session for this homework set will take place on March 12

Please choose a nickname (for publishing the results on the web) and sign one of your solutions both by the nickname and your real name; sign also all the other sheets either by your name or by your nickname.

Every exercise is worth 2 points, the starred one is a bonus exercise.

(1) Suppose we start with a bin with two balls, black and white. Repeatedly, take a random ball out of the bin and return it to the bin together with a new ball of the same color. After k rounds, there are $k + 2$ balls in the bin.

(a) For k odd, calculate the probability that there are more white balls than black balls in the bin.

(b) For an arbitrary k, calculate the probability that there are at most 10 white balls in the bin.

(2) Prove that $NP \subseteq coRP$ implies $NP = ZPP = RP = coRP = coNP$.

(3) Prove that $NP \subseteq BPP$ implies $NP = RP$.

(4) Let us play the following game. I write two distinct numbers on two cards; I shuffle the cards and put them in my hands (each of the two options has probability 1/2). You pick one of my hands and I give you the card in this hand. You can then choose to keep the card or swap it with the card in my other hand. If your card at the end has number x on it and my has y, you win $x - y$ and I lose that amount (equivalently, I win $y - x$). I.e., the amount of win is given in advance by the difference of the two numbers, and the winner is the player with the higher number at the end.

(a) Consider the following deterministic strategy: If you see a number larger than 21, you keep it, otherwise you swap. Determine the expected win depending on my choice of the two numbers.

(b) Propose a randomized strategy such that the expected win is always positive.

(4*) Assume that I choose the number as follows: For $i = 1, 2, \ldots$ I choose the pair $\{3^{i-1}, 3^i\}$ with probability 2^{-i}. Calculate for a given i and the situation that the first number is 3^i, the probability that the other number is 3^{i-1}, respectively 3^{i+1}. Based on this, for each i determine the expected win of the two deterministic strategies "always keep the first number" and "always swap". Explain as much as you can the overall choice of a good strategy and the expected outcome of the game.