1. Prove that for every matrix $A \in \mathbf{R}^{n \times m}$, the set of solutions to the system of equations $Ax = 0$ forms a subspace of $\mathbf{R}^m$.

   Find a basis of the space of solutions of the following system of equations.

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 0 \\ x_1 \qquad + 2x_3 \qquad + x_5 &= 0 \\ x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 &= 0 \end{aligned}$$

2. Write the definition of a *group*. Which of the following objects are groups, and why?

   - Integers with the operation of addition.
   - Integers with the operation of multiplication.
   - Bijective functions from $\mathbf{R}^2$ to $\mathbf{R}^2$ with the operation of composition.
   - Non-negative rational numbers with the operation of addition.
   - Positive rational numbers with the operation of multiplication.

3. Find some linear function $f : \mathbf{R}^4 \to \mathbf{R}^3$ such that $f(1,2,3,4) = (3,2,1)$ and $f(4,3,2,1) = (1,2,3)$, and write its matrix with respect to the standard bases of $\mathbf{R}^4$ and $\mathbf{R}^3$.

4. Write a definition of *kernel* and *image* of a linear function.

   Let $d : \mathcal{P}^n \to \mathcal{P}^n$ be the derivative on the space of polynomials of degree at most n, that is, the function defined by

   $$d(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \ldots + \alpha_n x^n) = \alpha_1 + 2\alpha_2 x + 3\alpha_3 x^2 + \ldots + n\alpha_n x^{n-1}$$

   for every $\alpha_0, \ldots, \alpha_n$. Determine $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$.

5. Let A be a square matrix.

   - Prove that if A has a right inverse (that is, a matrix B such that $AB = I$) and a left inverse (that is, a matrix C such that $CA = I$), then both inverses are equal ($B = C$).
   - Prove that A has a right inverse if and only if it has a left inverse.