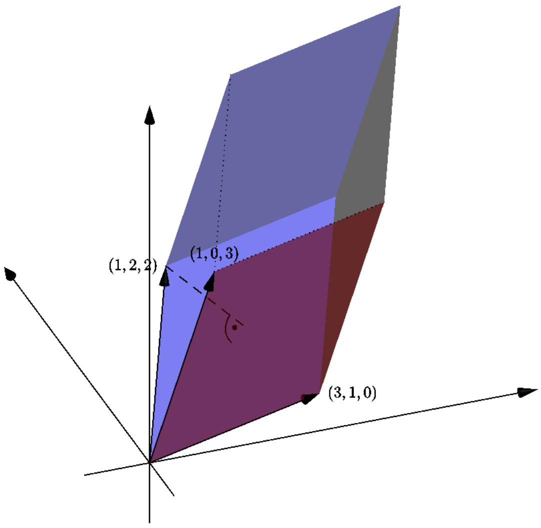
Determinants

Zdeněk Dvořák

March 23, 2015

Example 1. *Compute the volume of the parallelepiped with one vertex in* $(0,0,0)$ *and the incident edges given by vectors* $(1,2,2)$, $(1,0,3)$, $(3,1,0)$.



The volume of the parallelepiped is its height times the area of its base. The height is the distance of $(1,2,2)$ *from* $U = span((1,0,3),(3,1,0))$, *which is* $\frac{17}{\sqrt{91}}$ *(computed by taking the projection).*

Similarly, the area of the base (the parallelogram with sides $(1,0,3)$ *and* $(3,1,0)$*) is its height (the distance of* $(1,0,3)$ *from* $(3,1,0)$, *which is* $\sqrt{\frac{91}{10}}$*) times the length of its side* $(3,1,0)$ *(which is* $\sqrt{10}$*).*

Hence, the volume is

$$\frac{17}{\sqrt{91}} \cdot \sqrt{\frac{91}{10}} \cdot \sqrt{10} = 17.$$

For vectors $v_1, \ldots, v_n \in \mathbf{R}^n$, let $\mathrm{Vol}(v_1, \ldots, v_n)$ denote the volume of the parallelepiped with one vertex in $(0,0,0)$ and the incident edges given by v_1, $\ldots$, v_n.

Lemma 1. *For any* $s \neq t$ *and* $\alpha \in \mathbf{R}$, *let* $v'_i = v_i$ *for* $i \neq s$ *and* $v'_s = v_s + \alpha v_t$. *Then* $Vol(v_1, \ldots, v_n) = Vol(v'_1, \ldots, v'_n)$.

Proof. Shifting v_s in the plane parallel to the base $\mathrm{span}(v_1, \ldots, v_{s-1}, v_{s+1}, \ldots, v_n)$ of the parallelepiped does not change the height of the parallelepiped. $\square$

Example 2. *Compute the volume of the parallelepiped with one vertex in* $(0, 0, 0)$ *and the incident edges given by vectors* $(1, 2, 2)$, $(1, 0, 3)$, $(3, 1, 0)$, *using Lemma 1.*

$$Vol\begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & 3 \\ 3 & 1 & 0 \end{pmatrix} = Vol\begin{pmatrix} 1 & 2 & 2 \\ 0 & -2 & 1 \\ 0 & -5 & -6 \end{pmatrix} = Vol\begin{pmatrix} 1 & 2 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -17/2 \end{pmatrix}$$

Now,

- *the length of* $(0, 0, -17/2)$ *is* $|-17/2| = 17/2$.
- *span*$((0, 0, -17/2))$ *is the* z*-axis, and thus the distance from* $(0, -2, 1)$ *to span*$((0, 0, -17/2))$ *is* $|-2| = 2$.
- *span*$((0, -2, 1), (0, 0, -17/2))$ *is the plane spanned by the* y*- and* z*-axes, and the distance of* $(1, 2, 2)$ *from it is* $|1| = 1$.

Hence, the volume is $|(-17/2) \cdot (-2) \cdot 1| = 17$.

Recall:

Definition 1. *Let* π *be a permutation of* n. *The* <u>*sign*</u> *sgn*(π) *of* π *is*

$$(-1)^{n - \text{number of cycles of } \pi}.$$

Definition 2. *Let* A *be an* $n \times n$ *matrix. The* <u>*determinant*</u> *of* A *is*

$$\det(A) = \sum_{\pi:\ \text{permutation of } \{1, \ldots, n\}} sgn(\pi) A_{1,\pi(1)} A_{2,\pi(2)} \ldots A_{n,\pi(n)}.$$

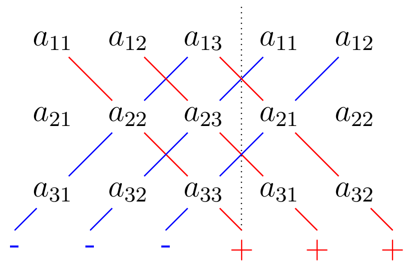
Example 3.

$$\det\begin{pmatrix} a_{11} \end{pmatrix} = a_{11}$$

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

Memorization help for 3×3 *matrices:*



Warning: similar scheme does not work for larger matrices!

Lemma 2. *If A is an upper triangular matrix,*

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{2n} \\ & & \dots & & \\ 0 & 0 & 0 & \dots & a_{nn} \end{pmatrix},$$

then $\det(A) = a_{11}a_{22}\dots a_{nn}$*. In particular,* $\det(I) = 1$*.*

Proof. Every other term in the definition of the determinant contains 0. □

Lemma 3. *Determinant is linear in each row. That is, let A, B, C, D be $n \times n$ matrices.*

- *If A is obtained from B by multiplying the r-th row by α, then* $\det(A) = \alpha \det(B)$*.*
- *If A differs from C and D only in the r-th row, and $A_{r,\star} = C_{r,\star} + D_{r,\star}$, then* $\det(A) = \det(C) + \det(D)$*.*

Proof.

- For $i \neq r$, we have $A_{i,\pi(i)} = B_{i,\pi(i)}$, and $A_{r,\pi(r)} = \alpha B_{r,\pi(r)}$. Hence,

$$\begin{aligned} \det(A) &= \sum_\pi \operatorname{sgn}(\pi) A_{1,\pi(1)} \dots A_{n,\pi(n)} \\ &= \alpha \sum_\pi \operatorname{sgn}(\pi) B_{1,\pi(1)} \dots B_{n,\pi(n)} \\ &= \det(B). \end{aligned}$$

- For $i \neq r$, we have $A_{i,\pi(i)} = C_{i,\pi(i)} = D_{i,\pi(i)}$, and $A_{r,\pi(r)} = C_{r,\pi(r)} +$

$D_{r,\pi(r)}$. Hence,

$$\begin{aligned}
\det(A) &= \sum_{\pi} \mathrm{sgn}(\pi) A_{1,\pi(1)} \dots A_{n,\pi(n)} \\
&= \sum_{\pi} \mathrm{sgn}(\pi) A_{1,\pi(1)} \dots A_{r-1,\pi(r-1)} (C_{r,\pi(r)} + D_{r,\pi(r)}) A_{r+1,\pi(r+1)} \dots A_{n,\pi(n)} \\
&= \left(\sum_{\pi} \mathrm{sgn}(\pi) A_{1,\pi(1)} \dots A_{r-1,\pi(r-1)} C_{r,\pi(r)} A_{r+1,\pi(r+1)} \dots A_{n,\pi(n)} \right) \\
&\quad + \left(\sum_{\pi} \mathrm{sgn}(\pi) A_{1,\pi(1)} \dots A_{r-1,\pi(r-1)} D_{r,\pi(r)} A_{r+1,\pi(r+1)} \dots A_{n,\pi(n)} \right) \\
&= \left(\sum_{\pi} \mathrm{sgn}(\pi) C_{1,\pi(1)} \dots C_{n,\pi(n)} \right) + \left(\sum_{\pi} \mathrm{sgn}(\pi) D_{1,\pi(1)} \dots D_{n,\pi(n)} \right) \\
&= \det(C) + \det(D).
\end{aligned}$$

□

Lemma 4. *Let A be an $n \times n$ matrix. If two of the rows of A are the same, then* $\det(A) = 0$.

Proof. Suppose that $A_{r,\star} = A_{s,\star}$ for some $r < s$. For any permutation π of $\{1, \dots, n\}$, let π' be the permutation such that $\pi'(r) = \pi(s)$, $\pi'(s) = \pi(r)$, and $\pi'(i) = \pi(i)$ for $i \neq r, s$. Then $A_{1,\pi(1)} A_{2,\pi(2)} \dots A_{n,\pi(n)} = A_{1,\pi'(1)} A_{2,\pi'(2)} \dots A_{n,\pi'(n)}$ and $\mathrm{sgn}(\pi) = -\mathrm{sgn}(\pi')$, and thus the contributions of π and π' to $\det(A)$ cancel each other. □

Lemma 5. *Let A, B be $n \times n$ matrices, let $s \neq t$, and let $\alpha \in \mathbf{R}$. If B is obtained from A by adding α times the s-th row to the t-th row (i.e., $B = A + \alpha e_t^T A_{s,\star}$), then* $\det(A) = \det(B)$.

Proof. If $\alpha = 0$, then $B = A$ and the claim is trivial. Let A_1 be obtained from A by multiplying the s-th row by α, $\det(A_1) = \alpha \det(A)$. Let A_2 be obtained from A_1 by adding the s-th row to the t-th row, and let A_1' be obtained from A_1 by replacing the t-th row by the s-th row. Note that A_1' has two equal rows, and thus $\det(A_1') = 0$ and $\det(A_2) = \det(A_1) + \det(A_1') = \det(A_1) = \alpha \det(A)$. Note that B is obtained from A_2 by multiplying the s-th row by $1/\alpha$, $\det(B) = \det(A_2)/\alpha = \det(A)$. □

Corollary 6. *For any $n \times n$ matrix A, we have*

$$Vol(A_{1,\star}, A_{2,\star}, \dots, A_{n,\star}) = |\det(A)|.$$

In particular, if the vectors $A_{1,\star}$, $\dots$, $A_{n,\star}$ have integral coordinates, then the volume of the corresponding parallelepiped is an integer.

Example 4. *Compute the volume of the parallelepiped with one vertex in* $(0,0,0)$ *and the incident edges given by vectors* $(1,2,2)$*,* $(1,0,3)$*,* $(3,1,0)$*, using Corollary 6.*

$$\begin{aligned}\det\begin{pmatrix}1 & 2 & 2\\ 1 & 0 & 3\\ 3 & 1 & 0\end{pmatrix} &= 1\cdot 0\cdot 0 + 2\cdot 3\cdot 3 + 2\cdot 1\cdot 1 - 2\cdot 0\cdot 3 - 1\cdot 3\cdot 1 - 2\cdot 1\cdot 0\\ &= 17,\end{aligned}$$

and thus the volume is 17*.*

Lemma 7. *Let* A, B *be* $n\times n$ *matrices. If* B *is obtained from* A *by exchanging the* r*-th and the* s*-th row, then* $\det(A) = -\det(B)$*.*

Proof. Let A_1 be obtained from A by adding the s-th row to the r-th row, $\det(A_1) = \det(A)$. Let A_2 be obtained from A_1 by multiplying the s-th row by -1, $\det(A_2) = -\det(A_1) = -\det(A)$. Let A_3 be obtained from A_2 by adding the r-th row to the s-th row, $\det(A_3) = \det(A_2) = -\det(A)$. Observe that B is obtained from A_3 by subtracting the s-th row from the r-th row, and thus $\det(B) = \det(A_3) = -\det(A)$. □

Algorithm 1. *To compute* $\det(A)$*, apply Gaussian elimination, and keep track of the sign changes when exchanging rows. Return the product of the diagonal entries of the resulting matrix, with the appropriate sign.*

Alternatively, we may also allow multiplying rows by non-zero constants, but we have to keep track of the effect of these operations on the determinant.

Example 5.

$$\begin{aligned}\det\begin{pmatrix}3 & 2 & 1\\ 1 & 0 & 3\\ 1 & 3 & 3\end{pmatrix} &= -\det\begin{pmatrix}1 & 0 & 3\\ 3 & 2 & 1\\ 1 & 3 & 3\end{pmatrix} = -\det\begin{pmatrix}1 & 0 & 3\\ 0 & 2 & -8\\ 0 & 3 & 0\end{pmatrix}\\ &= -2\det\begin{pmatrix}1 & 0 & 3\\ 0 & 1 & -4\\ 0 & 3 & 0\end{pmatrix} = -2\det\begin{pmatrix}1 & 0 & 3\\ 0 & 1 & -4\\ 0 & 0 & 12\end{pmatrix}\\ &= -24.\end{aligned}$$