Orthogonal and orthonormal sets

Zdeněk Dvořák

February 24, 2015

Definition 1. *A set $S \subseteq \mathbf{V}$ is orthogonal if $u \perp v$ for all distinct $u, v \in S$, and it is orthonormal if additionally $\|u\| = 1$ for every $u \in S$.*

Lemma 1. *Let $\mathbf{V}$ be an inner product space over field $\mathbf{F}$. If $S \subseteq \mathbf{V}$ is orthogonal and $o \notin S$, then S is linearly independent. In particular, any orthonormal set is linearly independent.*

Proof. Let $v_1, \ldots, v_n \in S$ and $\alpha_1, \ldots, \alpha_n \in \mathbf{F}$ satisfy

$$\alpha_1 v_1 + \ldots + \alpha_n v_n = o.$$

For $i = 1, \ldots, n$, we have

$$\begin{aligned} 0 = \langle o, v_i \rangle &= \langle \alpha_1 v_1 + \ldots + \alpha_n v_n, v_i \rangle \\ &= \alpha_1 \langle v_1, v_i \rangle + \ldots + \alpha_n \langle v_n, v_i \rangle \\ &= \alpha_i \langle v_i, v_i \rangle, \end{aligned}$$

since $\langle v_j, v_i \rangle = 0$ for $j \neq i$. Since $v_i \neq o$, we have $\langle v_i, v_i \rangle > 0$, and thus $\alpha_i = 0$ for every i.

Note that if S is orthonormal, then $o \notin S$, since $\|o\| = 0 \neq 1$. □

Theorem 2 (Properties of orthonormal bases)**.** *Let $\mathbf{V}$ be an inner product space and let $B = v_1, \ldots, v_n$ be an orthonormal basis of $\mathbf{V}$.*

1. *The coordinates of a vector v with respect to B are $(\langle v, v_1 \rangle, \langle v, v_2 \rangle, \ldots, \langle v, v_n \rangle)$.*

2. *If the coordinates of $u, v \in \mathbf{V}$ with respect to B are $(\alpha_1, \ldots, \alpha_n)$ and $(\beta_1, \ldots, \beta_n)$, respectively, then $\langle u, v \rangle = \alpha_1 \overline{\beta_1} + \ldots + \alpha_n \overline{\beta_n}$.*

3. *If the coordinates of $v \in \mathbf{V}$ with respect to B are $(\beta_1, \ldots, \beta_n)$, then $\|v\| = \sqrt{|\beta_1|^2 + \ldots + |\beta_n|^2}$.*

Proof. 1. Let the coordinates of v be $(\beta_1, \ldots, \beta_n)$, so that

$$v = \beta_1 v_1 + \ldots + \beta_n v_n.$$

For $i = 1, \ldots, n$, we have

$$\begin{aligned}
\langle v, v_i \rangle &= \langle \beta_1 v_1 + \ldots + \beta_n v_n, v_i \rangle \\
&= \beta_1 \langle v_1, v_i \rangle + \beta_2 \langle v_2, v_i \rangle + \ldots + \beta_n \langle v_n, v_i \rangle \\
&= \beta_i,
\end{aligned}$$

since $\langle v_j, v_i \rangle = 0$ if $i \neq j$ and $\langle v_i, v_i \rangle = 1$.

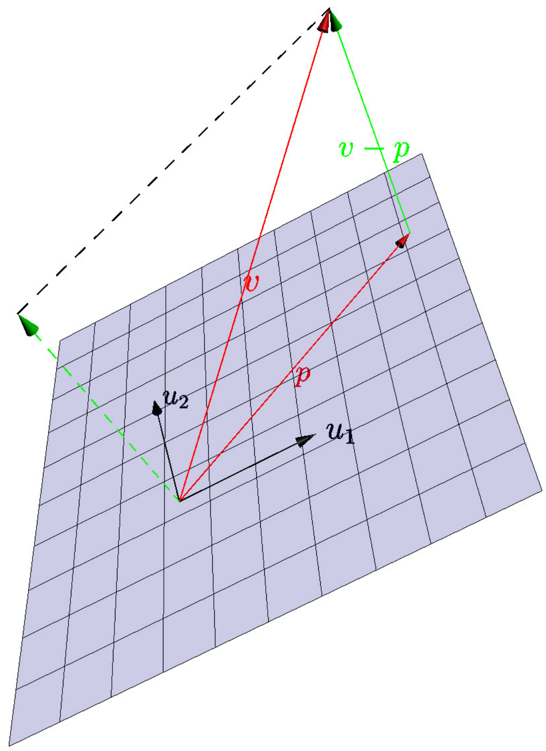
2. We have

$$\begin{aligned}
\langle u, v \rangle &= \langle \alpha_1 v_1 + \ldots + \alpha_n v_n, v \rangle \\
&= \alpha_1 \langle v_1, v \rangle + \ldots + \alpha_n \langle v_n, v \rangle \\
&= \alpha_1 \overline{\langle v, v_1 \rangle} + \ldots + \alpha_n \overline{\langle v, v_n \rangle} \\
&= \alpha_1 \overline{\beta_1} + \ldots + \alpha_n \overline{\beta_n}
\end{aligned}$$

3. We have

$$\begin{aligned}
\|v\| &= \sqrt{\langle v, v \rangle} \\
&= \sqrt{\beta_1 \overline{\beta_1} + \ldots + \beta_n \overline{\beta_n}} \\
&= \sqrt{|\beta_1|^2 + \ldots + |\beta_n|^2}
\end{aligned}$$

□



Algorithm 1 (Gram-Schmidt process)**.** *Let* $\mathbf{V}$ *be an inner product space.*

Input: *Vectors* $v_1, \ldots, v_n \in \mathbf{V}$

Output: *Vectors* $u_1, \ldots, u_m \in \mathbf{V}$

Let $m := 0$*. For* $k = 1, \ldots, n$*:*

- *Let* $v'_k = v_k - (\langle v_k, u_1\rangle\, u_1 + \ldots + \langle v_k, u_m\rangle\, u_m)$
- *If* $v'_k \neq o$*, then let* $m := m+1$ *and* $u_m = \frac{v'_k}{\|v'_k\|}$.

Example 1. *Gram-Schmidt process for vectors* $(1,1,0), (1,0,1), (3,2,1), (0,1,1)$*.*

$$
\begin{array}{lll}
v'_1 = (1,1,0) & \|v'_1\| = \sqrt{2} & u_1 = \dfrac{\sqrt{2}}{2}(1,1,0) \\
v'_2 = \dfrac{1}{2}(1,-1,2) & \|v'_2\| = \dfrac{\sqrt{6}}{2} & u_2 = \dfrac{\sqrt{6}}{6}(1,-1,2) \\
v'_3 = (0,0,0) & & \\
v'_4 = \dfrac{1}{3}(-2,2,2) & \|v'_4\| = \dfrac{2\sqrt{3}}{3} & u_3 = \dfrac{\sqrt{3}}{6}(-2,2,2)
\end{array}
$$

The result is $\frac{\sqrt{2}}{2}(1,1,0)$*,* $\frac{\sqrt{6}}{6}(1,-1,2)$*,* $\frac{\sqrt{3}}{6}(-2,2,2)$*.*

Theorem 3. *Let* $\mathbf{V}$ *be an inner product space, let* $S = v_1, \ldots, v_n$ *be a sequence of vectors of* $\mathbf{V}$*, and let* $T = u_1, \ldots, u_m$ *be the result of the Gram-Schmidt process applied to* S*. Then* T *is an orthonormal set and span*$(T) =$ *span*(S)*.*

Proof. We prove the claim by induction on n. For $n = 0$, the claim is trivially true, and thus assume that $n \geq 1$. Let $u_1, \ldots, u_{m'}$ be the result of the Gram-Schmidt process applied to $v_1, \ldots, v_{n-1}$, where m' is either m or $m - 1$. By the induction hypothesis, $u_1, \ldots, u_{m'}$ are othonormal and $\operatorname{span}(u_1, \ldots, u_{m'}) = \operatorname{span}(v_1, \ldots, v_{n-1})$.

Suppose first that $v_n \in \operatorname{span}(v_1, \ldots, v_{n-1}) = \operatorname{span}(u_1, \ldots, u_{m'})$. By Theorem 2, we have $v_n = \langle v_n, u_1 \rangle u_1 + \ldots + \langle v_n, u_{m'} \rangle u_{m'}$, and thus $v'_n = o$, $m = m'$, and $T = u_1, \ldots, u_{m'}$. Observe that

$$\operatorname{span}(T) = \operatorname{span}(u_1, \ldots, u_{m'}) = \operatorname{span}(v_1, \ldots, v_{n-1}) = \operatorname{span}(S).$$

Next, suppose that $v_n \notin \operatorname{span}(v_1, \ldots, v_{n-1}) = \operatorname{span}(u_1, \ldots, u_{m'})$. Since $\langle v_n, u_1 \rangle u_1 + \ldots + \langle v_n, u_{m'} \rangle u_{m'} \in \operatorname{span}(u_1, \ldots, u_{m'})$, it follows that $v'_n \neq o$, and thus $m = m' + 1$. Note that

- $\|u_m\| = \left\| \frac{v'_n}{\|v'_n\|} \right\| = \frac{\|v'_n\|}{\|v'_n\|} = 1$ and
- for $i = 1, \ldots, m'$,

$$\begin{aligned}
\langle v'_n, u_i \rangle &= \langle v_n - (\langle v_n, u_1 \rangle u_1 + \ldots + \langle v_n, u_{m'} \rangle u_{m'}), u_i \rangle \\
&= \langle v_n, u_i \rangle - (\langle v_n, u_1 \rangle \langle u_1, u_i \rangle + \ldots + \langle v_n, u_{m'} \rangle \langle u_{m'}, u_i \rangle) \\
&= \langle v_n, u_i \rangle - \langle v_n, u_i \rangle \\
&= 0,
\end{aligned}$$

  since $u_1, \ldots, u_{m'}$ is orthonormal, and thus $\langle u_m, u_i \rangle = 0$.

Therefore, $u_1, \ldots, u_m$ is orthonormal.

Note that $u_m \in \operatorname{span}(u_1, \ldots, u_{m'}, v_n) = \operatorname{span}(S)$, and thus $\operatorname{span}(T) \subseteq \operatorname{span}(S)$. Furthermore, since T is orthonormal, we have $\dim(\operatorname{span}(T)) = m = m' + 1$, and since $v_n \notin \operatorname{span}(v_1, \ldots, v_{n-1})$, we have $\dim(\operatorname{span}(S)) = 1 + \dim(\operatorname{span}(v_1, \ldots, v_{n-1})) = 1 + \dim(\operatorname{span}(u_1, \ldots, u_{m'})) = m' + 1$, hence $\dim(\operatorname{span}(S)) = \dim(\operatorname{span}(T))$ and $\operatorname{span}(S) = \operatorname{span}(T)$. $\square$

Corollary 4. *Every inner product space* $\mathbf{V}$ *of finite dimension has an orthonormal basis. Furthermore, if* $S \subset \mathbf{V}$ *is orthonormal, then there exists an orthonormal basis* B *of* $\mathbf{V}$ *such that* $S \subseteq B$*.*

Proof. Let $S = u_1, \ldots, u_k$. We can extend S to a basis $S' = u_1, \ldots, u_k, v_{k+1}, \ldots, v_n$ of $\mathbf{V}$. By the Gram-Schmidt process, we obtain an orthonormal basis $B = u_1, \ldots, u_k, w_{k+1}, \ldots, w_n$ (the process does not change u_1, $\ldots$, u_k, since they are orthonormal). $\square$