Dot product and inner product

Zdeněk Dvořák

February 24, 2015

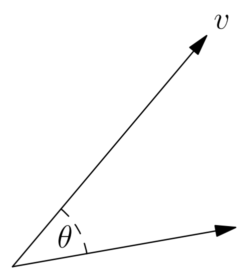
1 Dot (scalar) product of real vectors

Definition 1. *Let $u = (\alpha_1, \ldots, \alpha_n)$ and $v = (\beta_1, \ldots, \beta_n)$ be vectors from $\mathbf{R}^n$. The dot product of u and v is*

$$u \cdot v = \alpha_1\beta_1 + \alpha_2\beta_2 + \ldots + \alpha_n\beta_n.$$

Definition 2. *The Euclidean norm of $v = (\alpha_1, \ldots, \alpha_n) \in \mathbf{R}^n$ is*

$$|v| = \sqrt{\alpha_1^2 + \alpha_2^2 + \ldots + \alpha_n^2} = \sqrt{v \cdot v}.$$



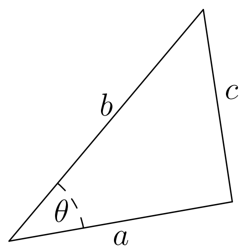
Lemma 1 (Geometric interpretation). *For any $u, v \in \mathbf{R}^n$ such that the angle between u and v is θ,*

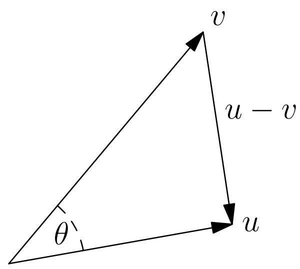
$$u \cdot v = |u||v| \cos\theta.$$

Proof. Note that the dot product is commutative and linear in both arguments, and thus

$$\begin{aligned}(u - v) \cdot (u - v) &= u \cdot (u - v) - v \cdot (u - v)\\ &= (u \cdot u - u \cdot v) - (v \cdot u - v \cdot v)\\ &= u \cdot u + v \cdot v - 2u \cdot v\end{aligned}$$

Recall that in a triangle





we have

$$c^2 = a^2 + b^2 - 2ab\cos\theta,$$

and thus

$$|u - v|^2 = |u|^2 + |v|^2 - 2|u||v|\cos\theta.$$

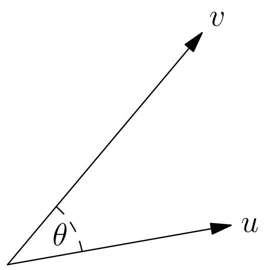
It follows that

$$\begin{aligned}
|u||v|\cos\theta &= \frac{|u|^2 + |v|^2 - |u - v|^2}{2} \\
&= \frac{u \cdot u + v \cdot v - (u - v) \cdot (u - v)}{2} \\
&= \frac{u \cdot u + v \cdot v - (u \cdot u + v \cdot v - 2u \cdot v)}{2} \\
&= \frac{2u \cdot v}{2} \\
&= u \cdot v.
\end{aligned}$$

□

Uses of dot product:

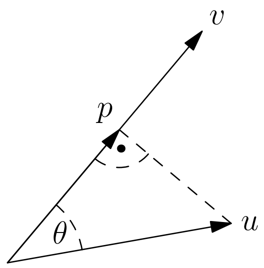
- Determining the angle between two vectors:



$$\theta = \arccos\frac{u \cdot v}{|u||v|}$$

- Two vectors are perpendicular iff their dot product is 0.

- Orthogonal projection:



The projection p of u on v (where θ is the angle between u and v) has norm

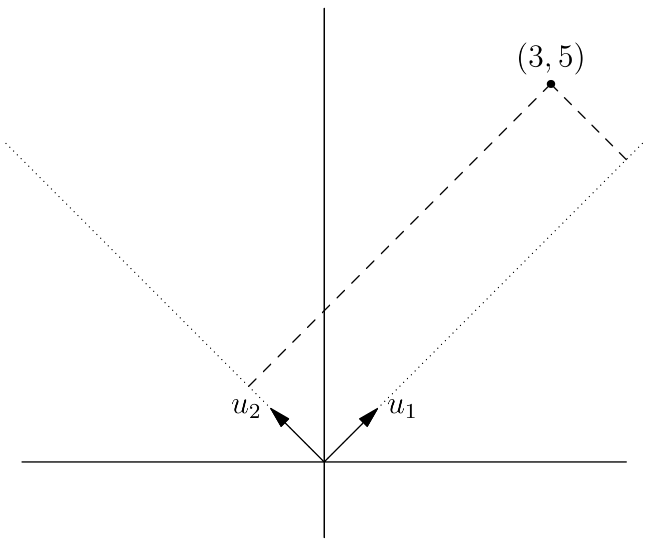
$$|u| \cos\theta = \frac{u \cdot v}{|v|}$$

and the same direction as v, hence

$$p = \frac{v}{|v|} \cdot \frac{u \cdot v}{|v|} = \frac{u \cdot v}{v \cdot v} v.$$

- Determining coordinates in an orthogonal basis (projections to basis vectors).

Example 1. *Let* $u_1 = (\sqrt{2}/2, \sqrt{2}/2)$ *and* $u_2 = (-\sqrt{2}/2, \sqrt{2}/2)$*. Determine the coordinates of* $(3, 5)$ *with respect to the basis* $B = u_1, u_2$*.*



Note that $u_1 \cdot u_2 = 0$ *(the vectors* u_1 *and* u_2 *are perpendicular) and* $|u_1| = |u_2| = 1$*. Hence, the coordinates are*

$$(3, 5) \cdot u_1 = 4\sqrt{2}$$

and

$$(3,5) \cdot u_2 = \sqrt{2}$$

2 Inner product spaces

Recall:

- $\mathbf{R}$: the field of real numbers
- $\mathbf{C}$: the field of complex numbers
- complex conjugation:
  - $\overline{\alpha + \beta i} = \alpha - \beta i$
  - $\overline{x + y} = \overline{x} + \overline{y}$
  - $\overline{xy} = \overline{x}\,\overline{y}$
  - $x\overline{x} = |x|^2$, where $|\alpha + \beta i| = \sqrt{\alpha^2 + \beta^2}$

Definition 3. *Let* $\mathbf{F}$ *be either* $\mathbf{R}$ *or* $\mathbf{C}$.

Inner product space is a vector space $\mathbf{V}$ *over* $\mathbf{F}$*, together with an inner product*

$$\langle \cdot, \cdot \rangle : \mathbf{V}^2 \to \mathbf{F}$$

satisfying the following axioms:

positive definiteness *For all* $v \in \mathbf{V}$, $\langle v, v \rangle$ *is a non-negative real number, and* $\langle v, v \rangle = 0$ *if and only if* $v = o$.

linearity in the first argument *For all* $u, v, w \in \mathbf{V}$ *and* $\alpha \in \mathbf{F}$,

$$\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$$
$$\langle \alpha u, w \rangle = \alpha \langle u, w \rangle$$

conjugate commutativity *For all* $u, v \in \mathbf{V}$,

$$\langle u, v \rangle = \overline{\langle v, u \rangle}.$$

Remark:

- $\langle o, v \rangle = 0 = \langle v, o \rangle$ for every $v \in \mathbf{V}$.
- If $\mathbf{F} = \mathbf{R}$, then

- the last axiom states commutativity $\langle u, v \rangle = \langle v, u \rangle$, and
- $\langle \cdot, \cdot \rangle$ is linear in the second argument as well

$$\langle w, u + v \rangle = \langle w, u \rangle + \langle w, v \rangle$$
$$\langle w, \alpha u \rangle = \alpha \langle w, u \rangle$$

- If $\mathbf{F} = \mathbf{C}$, then

$$\begin{aligned}
\langle w, u + v \rangle &= \overline{\langle u + v, w \rangle} \\
&= \overline{\langle u, w \rangle + \langle v, w \rangle} \\
&= \overline{\langle u, w \rangle} + \overline{\langle v, w \rangle} \\
&= \langle w, u \rangle + \langle w, v \rangle \\
\langle w, \alpha u \rangle &= \overline{\langle \alpha u, w \rangle} \\
&= \overline{\alpha \langle u, w \rangle} \\
&= \overline{\alpha} \overline{\langle u, w \rangle} \\
&= \overline{\alpha} \langle w, u \rangle .
\end{aligned}$$

  - $\langle \cdot, \cdot \rangle$ is <u>not</u> linear in the second argument, because of the conjugation in scalar multiplication.

Example 2.

- *Dot product gives an inner product on* $\mathbf{R}^n$*.*
- *Another example of possible inner product on* $\mathbf{R}^2$*:*

$$\langle (\alpha_1, \alpha_2), (\beta_1, \beta_2) \rangle = 2\alpha_1\beta_1 + \alpha_2\beta_2 - \alpha_1\beta_2 - \alpha_2\beta_1$$

  - *positive definiteness:* $\langle (\alpha_1, \alpha_2), (\alpha_1, \alpha_2) \rangle = \alpha_1^2 + (\alpha_1 - \alpha_2)^2 \geq 0$, *and equal to* 0 *if and only if* $\alpha_1 = 0$ *and* $\alpha_1 - \alpha_2 = 0 \Rightarrow \alpha_2 = 0$.

- *Complex dot product on* $\mathbf{C}^n$*:*

$$(\alpha_1, \ldots, \alpha_n) \cdot (\beta_1, \ldots, \beta_n) = \alpha_1\overline{\beta_1} + \ldots + \alpha_n\overline{\beta_n}.$$

- *Standard inner product on the space of continuous functions* $f : [\alpha, \beta] \to \mathbf{R}$*:*

$$\langle f, g \rangle = \int_\alpha^\beta f(x)g(x)dx$$

Definition 4. *Let* $\mathbf{V}$ *be an inner product space. Vectors* $u, v \in \mathbf{V}$ *are* *orthogonal* *if* $\langle u, v \rangle = 0$*. We write* $u \perp v$*.*

Example 3.

- $(1, 0)$ *and* $(0, 1)$ *are orthogonal with respect to the dot product, since* $(1, 0) \cdot (0, 1) = 0$*.*
- $(1, 0)$ *and* $(0, 1)$ *are* *not* *orthogonal with respect to the inner product*

$$\langle (\alpha_1, \alpha_2), (\beta_1, \beta_2) \rangle = 2\alpha_1\beta_1 + \alpha_2\beta_2 - \alpha_1\beta_2 - \alpha_2\beta_1,$$

  since $\langle (1, 0), (0, 1) \rangle = -1$*.*

- $f(x) = \sin x$ *and* $g(x) = 1$ *are orthogonal with respect to the standard inner product on the space of continuous functions from* $[0, 2\pi]$*:*

$$\int_0^{2\pi} f(x)g(x)dx = \int_0^{2\pi} \sin x \, dx = [-\cos x]_0^{2\pi} = \cos 0 - \cos(2\pi) = 0.$$

Theorem 2 (Pythagoras theorem)**.** *Let* $\mathbf{V}$ *be an inner product space and let* $u, v \in \mathbf{V}$*. If* $u \perp v$*, then*

$$\langle u, u \rangle + \langle v, v \rangle = \langle u + v, u + v \rangle .$$

Proof.

$$\begin{aligned} \langle u + v, u + v \rangle &= \langle u, u \rangle + \langle v, v \rangle + \langle u, v \rangle + \langle v, u \rangle \\ &= \langle u, u \rangle + \langle v, v \rangle , \end{aligned}$$

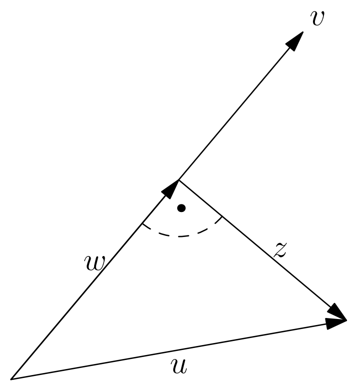
since $\langle u, v \rangle = 0$ and $\langle v, u \rangle = \overline{\langle u, v \rangle} = 0$. □

Theorem 3 (Cauchy-Schwarz inequality)**.** *Let* $\mathbf{V}$ *be an inner product space. Then for all* $u, v \in \mathbf{V}$*,*

$$| \langle u, v \rangle |^2 \leq \langle u, u \rangle \langle v, v \rangle ,$$

and if u *and* v *are linearly independent, then the inequality is sharp.*

Proof. The claim is clearly true if $v = o$, hence assume that $\langle v, v \rangle > 0$.



Let $w = \frac{\langle u,v \rangle}{\langle v,v \rangle} v$ and $z = u - w$. Then

$$\begin{aligned}
\langle z, v \rangle &= \left\langle u - \frac{\langle u,v \rangle}{\langle v,v \rangle} v, v \right\rangle \\
&= \langle u, v \rangle - \left\langle \frac{\langle u,v \rangle}{\langle v,v \rangle} v, v \right\rangle \\
&= \langle u, v \rangle - \frac{\langle u,v \rangle}{\langle v,v \rangle} \langle v, v \rangle \\
&= 0,
\end{aligned}$$

and thus $v \perp z$ and $w \perp z$.

Since $u = w + z$, Pythagoras theorem implies

$$\begin{aligned}
\langle u, u \rangle &= \langle w, w \rangle + \langle z, z \rangle \\
&\geq \langle w, w \rangle \\
&= \frac{\langle u,v \rangle}{\langle v,v \rangle} \overline{\frac{\langle u,v \rangle}{\langle v,v \rangle}} \langle v, v \rangle \\
&= \left| \frac{\langle u,v \rangle}{\langle v,v \rangle} \right|^2 \langle v, v \rangle \\
&= \frac{|\langle u, v \rangle|^2}{\langle v, v \rangle},
\end{aligned}$$

and thus

$$\langle u, u \rangle \langle v, v \rangle \geq |\langle u, v \rangle|^2.$$

The equality holds only if $z = o$, i.e., if $u = \frac{\langle u,v \rangle}{\langle v,v \rangle} v$, which implies that u and v are linearly dependent. $\square$

Example 4. *Let $x_1, \ldots, x_n$ be positive real numbers. Prove that*

$$x_1^2 + \ldots + x_n^2 \geq \frac{(x_1 + \ldots + x_n)^2}{n},$$

where the equality holds if and only if $x_1 = x_2 = \ldots = x_n$.

Proof. We apply the Cauchy-Schwarz inequality for the dot product of $u = (x_1, \ldots, x_n)$ and $v = (1, \ldots, 1)$:

$$\begin{aligned}(x_1^2 + \ldots + x_n^2)n &= (u \cdot u)(v \cdot v) \\ &\geq (u \cdot v)^2 \\ &= (x_1 + \ldots + x_n)^2,\end{aligned}$$

where the equality only holds if u and v are linearly dependent, i.e., $x_1 = \ldots = x_n$. □

Definition 5. *Let* $\mathbf{V}$ *be a vector space over a field* $\mathbf{F} \in \{\mathbf{R}, \mathbf{C}\}$. *A function* $s : \mathbf{V} \to \mathbf{R}$ *is a norm if*

- $s(v) \geq 0$ *for every* $v \in \mathbf{V}$, *and* $s(v) = 0$ *if and only if* $v = o$.
- $s(\alpha v) = |\alpha| s(v)$ *for every* $v \in \mathbf{V}$ *and* $\alpha \in \mathbf{F}$.
- $s(u + v) \leq s(u) + s(v)$ *for every* $u, v \in \mathbf{V}$ *(triangle inequality).*

Definition 6. *The norm induced by an inner product is*

$$\|v\| = \sqrt{\langle v, v \rangle}.$$

- If $\langle \cdot, \cdot \rangle$ is the dot product, then $\| \cdot \|$ is the Euclidean norm.
- Pythagoras theorem reformulated using the norm: if $u \perp v$, then

$$\|u\|^2 + \|v\|^2 = \|u + v\|^2$$

- Cauchy-Schwarz inequality reformulated using the norm:

$$|\langle u, v \rangle| \leq \|u\| \|v\|$$

- The triangle inequality holds because of Cauchy-Schwarz:

$$\begin{aligned}\|u + v\|^2 &= \langle u + v, u + v \rangle \\ &= \|u\|^2 + \|v\|^2 + \langle u, v \rangle + \langle v, u \rangle \\ &\leq \|u\|^2 + \|v\|^2 + 2|\langle u, v \rangle| \\ &\leq \|u\|^2 + \|v\|^2 + 2\|u\|\|v\| \\ &= (\|u\| + \|v\|)^2\end{aligned}$$