Reminders: linear functions

Let $\mathbf{U}$ and $\mathbf{V}$ be vector spaces over the same field $\mathbf{F}$.

Definition

A function $f : \mathbf{U} \to \mathbf{V}$ is linear if

- for every $u_1, u_2 \in \mathbf{U}$,

$$f(u_1 + u_2) = f(u_1) + f(u_2), \text{ and}$$

- for every $u \in \mathbf{U}$ and $\alpha \in \mathbf{F}$,

$$f(\alpha u) = \alpha f(u).$$

Reminders: linear functions

Let $B = u_1, \ldots, u_n$ be a basis of **U**, let C be a basis of **V**.

- Linear function is uniquely determined by its values on a basis.
- Columns of the matrix $[f]_{B,C}$ of the function are coordinates (w.r. to C) of $f(u_1), \ldots, f(u_n)$.
- $[f]_{B,C}[u]_B^T = [f(u)]_C^T$

Reminders: matrices of linear functions

Let **U**, **V**, and **W** be vector spaces over the same field **F**, with bases $B = u_1, \dots, u_n$, C, and D, respectively.

Lemma

For any linear $f : \mathbf{U} \to \mathbf{V}$ *and* $g : \mathbf{V} \to \mathbf{W}$*,*

$$[gf]_{B,D} = [g]_{C,D}[f]_{B,C}.$$

Reminders: isomorphism

Definition

A linear function $f : \mathbf{U} \to \mathbf{V}$ is an isomorphism if f is bijective (1-to-1 and onto).

Lemma

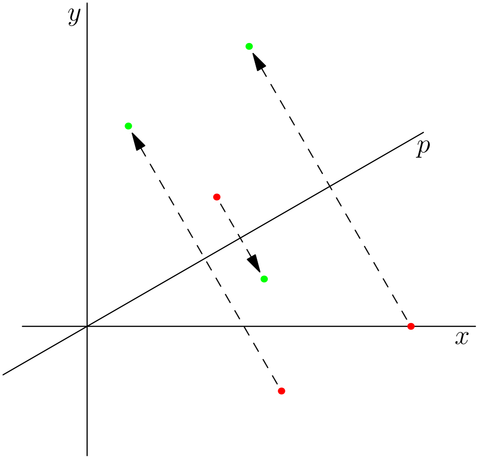
If $f : \mathbf{U} \to \mathbf{V}$ is an isomorphism, then f^{-1} is an isomorphism and

$$[f^{-1}]_{C,B} = [f]_{B,C}^{-1}.$$

Example: linear transformations of the plane

Problem

Let p be the line in $\mathbf{R}^2$ *through the origin in* 30 *degrees angle. To which point is* (x, y) *mapped by reflection across the p axis?*



Example: linear transformations of the plane

Problem

Let p be the line in $\mathbf{R}^2$ *through the origin in* 30 *degrees angle. To which point is* (x, y) *mapped by reflection across the p axis?*

- The reflection across the p axis defines an isomorphism $g : \mathbf{R}^2 \to \mathbf{R}^2$.
- Let r be the rotation by 30 degrees.
- Let f be the reflection across the x axis.
- $g = rfr^{-1}$, hence
- $[g] = [r][f][r]^{-1}$ with respect to the standard basis.

Example: linear transformations of the plane

Problem

Let p be the line in $\mathbf{R}^2$ through the origin in 30 degrees angle. To which point is (x, y) mapped by reflection across the p axis?

- r: the rotation by 30 degrees.
- f: the reflection across the x axis.

$$r(1,0) = (\sqrt{3}/2, 1/2) \qquad f(1,0) = (1,0)$$

$$r(0,1) = (-1/2, \sqrt{3}/2) \qquad f(0,1) = (0,-1)$$

$$[r] = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix} \qquad [f] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Example: linear transformations of the plane

Problem

Let p be the line in $\mathbf{R}^2$ *through the origin in* 30 *degrees angle. To which point is* (x, y) *mapped by reflection across the p axis?*

$$[g] = [r][f][r]^{-1} = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$$

Hence, $g(x, y) = (x/2 + \sqrt{3}y/2, \sqrt{3}x/2 - y/2)$.

Example: composition of rotations

Let $r_\alpha : \mathbf{R}^2 \to \mathbf{R}^2$ be the rotation by angle α.

$$r_\alpha(1,0) = (\cos\alpha, \sin\alpha)$$
$$r_\alpha(0,1) = (-\sin\alpha, \cos\alpha)$$
$$[r_\alpha] = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

Note that $r_{\alpha+\beta} = r_\alpha r_\beta$, and $[r_{\alpha+\beta}] = [r_\alpha][r_\beta]$:

$$\begin{aligned} \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix} &= \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \\ &= \begin{pmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{pmatrix} \end{aligned}$$

Therefore,

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

Linear functions and independent sets

Let **U** and **V** be vector spaces over the same field **F**.

Lemma

If a linear function $f : \mathbf{U} \to \mathbf{V}$ is 1-to-1 and $\{u_1, \ldots, u_k\} \subseteq \mathbf{U}$ is an independent set, then $\{f(u_1), \ldots, f(u_k)\}$ is an independent set in $\mathbf{V}$.

Proof.

Suppose that $\alpha_1 f(u_1) + \ldots + \alpha_k f(u_k) = o$.

- Let $u = \alpha_1 u_1 + \ldots + \alpha_k u_k$.
- $f(u) = f(\alpha_1 u_1 + \ldots + \alpha_k u_k) = \alpha_1 f(u_1) + \ldots + \alpha_k f(u_k) = o$.
- Since f is 1-to-1, $f(u) = o$, and $f(o) = o$, we have $u = o$.
- Since $\{u_1, \ldots, u_k\}$ is linearly independent, $\alpha_1 = \ldots = \alpha_k = 0$.

$\square$

Linear functions and independent sets

Let $\mathbf{U}$ and $\mathbf{V}$ be vector spaces over the same field $\mathbf{F}$.

Lemma

If a linear function $f : \mathbf{U} \to \mathbf{V}$ is 1-to-1 and $\{u_1, \ldots, u_k\} \subseteq \mathbf{U}$ is an independent set, then $\{f(u_1), \ldots, f(u_k)\}$ is an independent set in $\mathbf{V}$.

Corollary

If a function $f : \mathbf{U} \to \mathbf{V}$ is an isomorphism and $u_1, \ldots, u_k$ is a basis of $\mathbf{U}$, then $f(u_1), \ldots, f(u_k)$ is a basis of $\mathbf{V}$.

Proof.

- $f(u_1), \ldots, f(u_k)$ is independent
- $f(u_1), \ldots, f(u_k), v$ is not independent for any $v \in \mathbf{V}$, since $u_1, \ldots, u_k, f^{-1}(v)$ is not independent.

$\square$

Isomorphic spaces

Definition

Two spaces **U** and **V** are isomorphic if there exists an isomorphism from **U** to **V** (and vice versa).

Examples:

- $\mathcal{P}_n$ and $\mathbf{R}^{n+1}$ are isomorphic via isomorphism mapping $p = \alpha_0 + \alpha_1 x + \ldots + \alpha_n x^n$ to $(\alpha_0, \alpha_1, \ldots, \alpha_n)$.
- $\mathcal{P}_n$ and $\mathbf{R}^{n+1}$ are also isomorphic via isomorphism mapping p to $(p(0), p(1), \ldots, p(n))$.

Corollary

Any two isomorphic spaces have the same dimension.

Isomorphism and coordinates

Let $\mathbf{V}$ be a vector space over the field $\mathbf{F}$.
Let B be a basis of $\mathbf{V}$.
Let $\mathrm{coord}_B : \mathbf{V} \to \mathbf{F}^{\dim \mathbf{V}}$ be defined by $\mathrm{coord}_B(v) = [v]_B$.

Lemma

coord_B is an isomorphism from $\mathbf{V}$ to $\mathbf{F}^{\dim \mathbf{V}}$

Corollary

Vector spaces over the same field are isomorphic if and only if they have the same dimension.

Spaces associated with linear functions

Let $f : \mathbf{U} \to \mathbf{V}$ be a linear function.

Definition

Image of f consists of all elements of $\mathbf{V}$ to that f maps something.

$$\mathrm{Im}(f) = \{f(u) : u \in \mathbf{U}\}$$

Kernel of f consists of all elements of $\mathbf{U}$ that f maps to o.

$$\mathrm{Ker}(f) = \{u \in \mathbf{U} : f(u) = o\}$$

Example

Let $f : \mathbf{R^3} \to \mathbf{R^2}$, $f(x, y, z) = (x, x)$.

- $\mathrm{Im}(f) = \{(x, x) : x \in \mathbf{R}\} = \mathrm{span}\{(1, 1)\}$
- $\mathrm{Ker}(f) = \{(0, y, z) : y, z \in \mathbf{R}\} = \mathrm{span}\{(0, 1, 0), (0, 0, 1)\}$

Image and kernel are subspaces

Let **U** and **V** be vector spaces over a field **F**.

Lemma

For any linear function $f : \mathbf{U} \to \mathbf{V}$, both Im$(f)$ and Ker(f) are vector spaces.

Proof.

- If $v_1, v_2 \in \mathrm{Im}(f)$, then $v_1 = f(u_1)$ and $v_2 = f(u_2)$ for some $u_1, u_2 \in \mathbf{U}$.
- $v_1 + v_2 = f(u_1) + f(u_2) = f(u_1 + u_2) \in \mathrm{Im}(f)$
- $\alpha v_1 = \alpha f(u_1) = f(\alpha u_1) \in \mathrm{Im}(f)$
- $o = f(o) \in \mathrm{Im}(f)$

$\square$

Image and kernel are subspaces

Let **U** and **V** be vector spaces over a field **F**.

Lemma

For any linear function $f : \mathbf{U} \to \mathbf{V}$, both Im$(f)$ and Ker(f) are vector spaces.

Proof.

- If $u_1, u_2 \in \mathrm{Ker}(f)$, then $f(u_1) = f(u_2) = o$.
- $f(u_1 + u_2) = f(u_1) + f(u_2) = o + o = o$
- $f(\alpha u_1) = \alpha f(u_1) = \alpha o = o$
- $f(o) = o$

□

Related matrix spaces

Let **F** be a field.

Definition

For an $n \times m$ matrix A with entries from **F**, let

$$\mathrm{Im}(A) = \{Ax : x \in \mathbf{F}^m\}$$

and

$$\mathrm{Ker}(A) = \{x \in \mathbf{F}^m : Ax = 0\}$$

Note:

- $\mathrm{Im}(A) = \mathrm{span}(A_{\star,1}, A_{\star,2}, \ldots, A_{\star,m}\} = \mathrm{Col}(A)$
- $\mathrm{Ker}(A)$ is the set of solutions of the system of linear equations $Ax = 0$.
- For $f : \mathbf{F}^m \to \mathbf{F}^n$, $f(x) = Ax$, we have
  - $\mathrm{Ker}(A) = \mathrm{Ker}(f)$
  - $\mathrm{Im}(A) = \mathrm{Im}(f)$,

  hence $\mathrm{Ker}(A)$, $\mathrm{Im}(A)$ are vector spaces.

Kernel and image of a matrix vs function

- Let $\mathbf{U}$ and $\mathbf{V}$ be vector spaces over the same field $\mathbf{F}$.
- Let B be a basis of $\mathbf{U}$, let C be a basis of $\mathbf{V}$.
- Let $\mathrm{coord}_B : \mathbf{U} \to \mathbf{F}^{\dim \mathbf{U}}$ be defined by $\mathrm{coord}_B(u) = [u]_B^T$.
- Let $\mathrm{coord}_C : \mathbf{V} \to \mathbf{F}^{\dim \mathbf{V}}$ be defined by $\mathrm{coord}_C(v) = [v]_C^T$.

Lemma

$Im([f]_{B,C})$ consists of coordinates (with respect to C) of $Im(f)$; i.e., coord_C is an isomorphism from $Im(f)$ to $Im([f]_{B,C})$.

$Ker([f]_{B,C})$ consists of coordinates (with respect to B) of $Ker(f)$; i.e., coord_B is an isomorphism from $Ker(f)$ to $Ker([f]_{B,C})$.

Example(1)

Problem

Let $f : \mathbf{R}^3 \to \mathbf{R}^3$ be defined by $f(x, y, z) = (x - y, y - z, z - x)$. Determine $Im(f)$ and $Ker(f)$.

With respect to the standard bases,

$$[f] = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}, \mathrm{RREF}([f]) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

- $\{v^T : v \in \mathrm{Im}(f)\} = \{[v]^T : v \in \mathrm{Im}(f)\} = \mathrm{Im}([f]) = \mathrm{Col}([f])$
- Basis column indices are 1, 2, hence the 1*st* and 2*nd* column of $[f]$ form a basis of $\mathrm{Col}([f])$.

$$\mathrm{Im}(f) = \mathrm{span}(\{(1, 0, -1), (-1, 1, 0)\}).$$

Example(1)

Problem

Let $f : \mathbf{R}^3 \to \mathbf{R}^3$ be defined by $f(x, y, z) = (x - y, y - z, z - x)$. Determine $Im(f)$ and $Ker(f)$.

With respect to the standard bases,

$$[f] = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}, \mathrm{RREF}([f]) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

- $\{u^T : u \in \mathrm{Ker}(f)\} = \{[u]^T : u \in \mathrm{Ker}(f)\} = \mathrm{Ker}([f])$.
- $\mathrm{Ker}([f])$ is the set of solutions of $[f]x = 0$
- the same as the set of solutions of $\mathrm{RREF}([f])x = 0$

$$\mathrm{Ker}(f) = \mathrm{span}(\{(1, 1, 1)\})$$

Example(2)

Problem

Let $f : \mathcal{P}_2 \to \mathbf{R}^2$ be defined by $f(p) = (p(0), p(2))$. Determine $Im(f)$ and $Ker(f)$.

Let $B = 1, x, x^2$ be a basis of $\mathcal{P}_2$, let $C = (1, 0), (0, 1)$.

$$[f]_{B,C} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 4 \end{pmatrix}, \mathrm{RREF}([f]_{B,C}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\mathrm{Im}([f]_{B,C}) = \mathrm{span}(\{(1, 1)^T, (0, 2)^T\})$$

hence

$$\mathrm{Im}(f) = \mathrm{span}(\{(1, 1), (0, 2)\}) = \mathbf{R}^2.$$

Example(2)

Problem

Let $f : \mathcal{P}_2 \to \mathbf{R}^2$ be defined by $f(p) = (p(0), p(2))$. Determine $Im(f)$ and $Ker(f)$.

Let $B = 1, x, x^2$ be a basis of $\mathcal{P}_2$, let $C = (1,0),(0,1)$.

$$[f]_{B,C} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 4 \end{pmatrix}, \mathrm{RREF}([f]_{B,C}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

- The set of solutions to $[f]_{B,C}x = 0$ is $\mathrm{span}(\{(0,-2,1)^T\})$.
- $(0,-2,1) = [x^2 - 2x]_B$

$$\mathrm{Ker}(f) = \mathrm{span}(\{x^2 - 2x\})$$

Dimensions of kernel and image

Let **U** and **V** be vector spaces over the same field **F**.

Lemma

For any linear function $f : \mathbf{U} \to \mathbf{V}$,

$$\dim Im(f) + \dim Ker(f) = \dim \mathbf{U}$$

Proof.

Let B be a basis of **U**, let C be a basis of **V**.

- It suffices to prove $\dim \mathrm{Im}([f]_{B,C}) + \dim \mathrm{Ker}([f]_{B,C}) = |B|$.
- $\dim \mathrm{Im}([f]_{B,C}) = \dim \mathrm{Col}([f]_{B,C}) = \mathrm{rank}([f]_{B,C})$
  - number of basis columns of $\mathrm{RREF}([f]_{B,C})$
- $\dim \mathrm{Ker}([f]_{B,C})$ is the dimension of the space of solutions of $[f]_{B,C}x = 0$
  - number of non-basis columns of $\mathrm{RREF}([f]_{B,C})$

$\square$

Kernel, image and 1-to-1 functions

Let $\mathbf{U}$ and $\mathbf{V}$ be vector spaces over the same field $\mathbf{F}$.

Lemma

For a linear function $f : \mathbf{U} \to \mathbf{V}$, the following are equivalent:

1. $Ker(f) = \{o\}$
2. *f is 1-to-1*
3. *For every independent set $\{u_1, \ldots, u_k\}$ in $\mathbf{U}$, the set $\{f(u_1), \ldots, f(u_k)\}$ is independent in $\mathbf{V}$.*
4. *For a basis $B = \{u_1, \ldots, u_k\}$ of $\mathbf{U}$, the set $\{f(u_1), \ldots, f(u_k)\}$ is independent in $\mathbf{V}$.*

Proof.

1 $\Rightarrow$ 2 If $f(x) = f(y)$, then $o = f(x) - f(y) = f(x - y)$, and thus $x - y \in \text{Ker}(f)$. Hence, $x - y = o$ and $x = y$.

$\square$

Let **U** and **V** be vector spaces over the same field **F**.

Lemma

For a linear function $f : \mathbf{U} \to \mathbf{V}$, the following are equivalent:

1. $Ker(f) = \{o\}$
2. *f is 1-to-1*
3. *For every independent set $\{u_1, \ldots, u_k\}$ in **U**, the set $\{f(u_1), \ldots, f(u_k)\}$ is independent in **V**.*
4. *For a basis $B = \{u_1, \ldots, u_k\}$ of **U**, the set $\{f(u_1), \ldots, f(u_k)\}$ is independent in **V**.*

Proof.

2 $\Rightarrow$ 3 Proved before.

$\square$

Kernel, image and 1-to-1 functions

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4. *For a basis $B = \{u_1, \dots, u_k\}$ of $\mathbf{U}$, the set $\{f(u_1), \dots, f(u_k)\}$ is independent in $\mathbf{V}$.*

Proof.

3 $\Rightarrow$ 4 Trivial.

$\square$

Kernel, image and 1-to-1 functions

Let $\mathbf{U}$ and $\mathbf{V}$ be vector spaces over the same field $\mathbf{F}$.

Lemma

For a linear function $f : \mathbf{U} \to \mathbf{V}$, the following are equivalent:

1. $Ker(f) = \{o\}$
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3. *For every independent set $\{u_1, \ldots, u_k\}$ in $\mathbf{U}$, the set $\{f(u_1), \ldots, f(u_k)\}$ is independent in $\mathbf{V}$.*
4. *For a basis $B = \{u_1, \ldots, u_k\}$ of $\mathbf{U}$, the set $\{f(u_1), \ldots, f(u_k)\}$ is independent in $\mathbf{V}$.*

Proof.

4 $\Rightarrow$ 1 Since $f(u_1), \ldots, f(u_k)$ is independent, $\dim \mathrm{Im}(f) \geq k$, and $\dim \mathrm{Ker}(f) \leq 0$.

$\square$