Linear Algebra I: basic information

Lecturer:

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- office hours: by appointment

Linear Algebra I: basic information

Grading:

- a “pass” grade (zápočet) from tutorials required before final exam
  - Morteza Monemizadeh is in charge of the tutorials and will give you more information
- a combined written + oral final exam
  - a sample exam will be available in December

Linear Algebra I: basic information

Study materials:

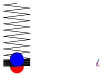
- lecture notes or slides posted at http://atrey.karlin.mff.cuni.cz/~rakdver/tea_lai_z15.html
- A First Course in Linear Algebra
- Matoušek: Thirty-three Miniatures: Mathematical and Algorithmic Applications of Linear Algebra

Why study linear algebra

Useful tool for many other branches of mathematics

- in physics: linear differential equations, Hilbert spaces, eigenvalues, ...
- in combinatorics: linear recurrences, proofs using rank, linear independence, ...

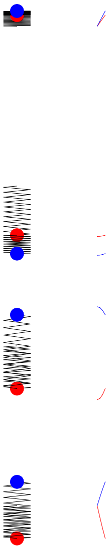
Why study linear algebra



$$y_1'' = -\frac{k_1 + k_2}{m_1} y_1 + \frac{k_2}{m_1} y_2$$

$$y_2'' = \frac{k_2}{m_2} y_1 - \frac{k_2}{m_2} y_2$$

Why study linear algebra



Why study linear algebra

In graphics and sound processing (Fourier transformation, . . .)

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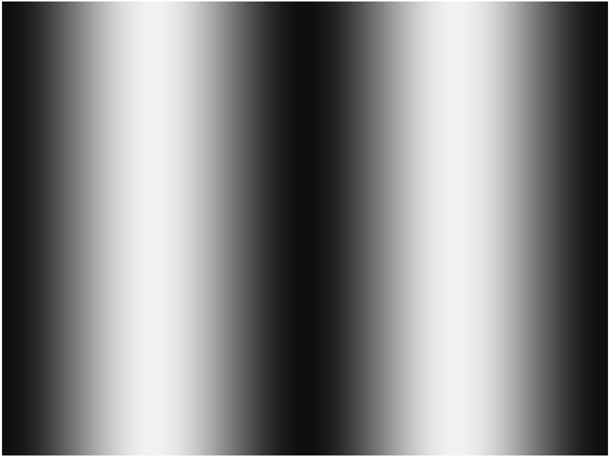

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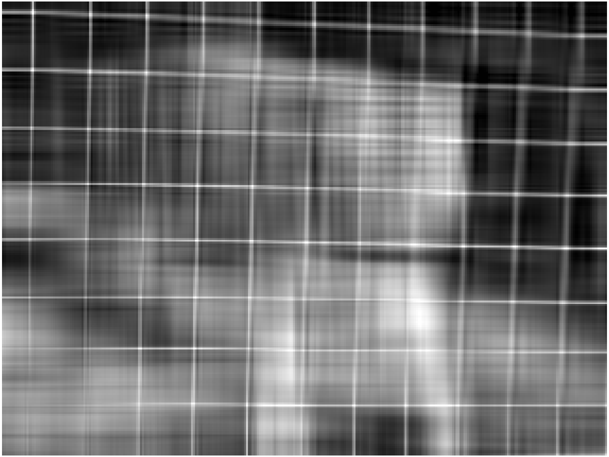
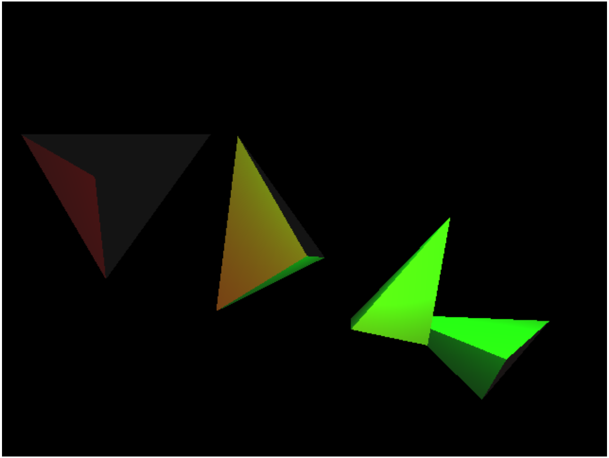


Why study linear algebra

In computer graphics



Data fitting

Measured values:

x	-2	-1.5	-1.0	-0.5	0.0	0.5	1	1.5
y	9.2	4.6	1.8	0.9	1.1	2.6	6	1.8

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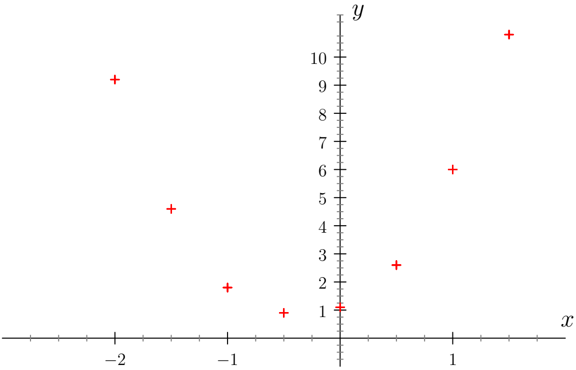

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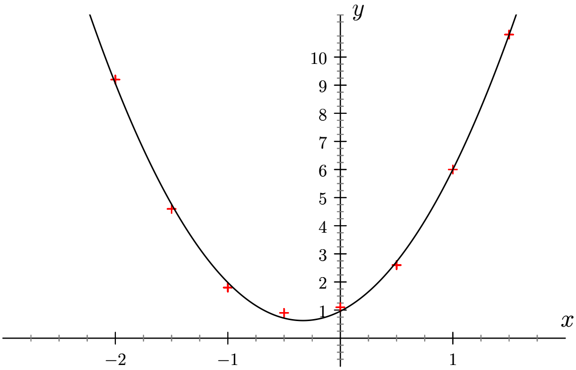
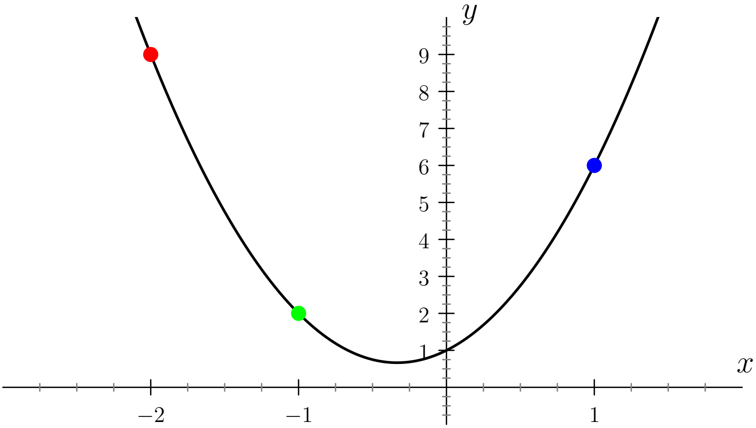


$$y \approx 3.03x^2 + 2.01x + 0.96$$

Easier example

Find the equation of quadratic function through points

$$(-2, 9),\ (-1, 2),\ \text{and}\ (1, 6)$$



Easier example

Find the equation of quadratic function through points

$$(-2, 9),\ (-1, 2),\ \text{and}\ (1, 6)$$

General equation:

$$y = ax^2 + bx + c$$

$$\begin{aligned} 9 &= 4a - 2b + c \\ 2 &= a - b + c \\ 6 &= a + b + c \end{aligned}$$

$$6 - 2 = (a + b + c) - (a - b + c) = 2b \Rightarrow b = 2$$

Hence, $c = 6 - a - b = 4 - a$, and

$$9 = 4a - 2b + c = 4a - 4 + (4 - a) = 3a$$

Consequently, $a = 3$ and $c = 4 - a = 1$.

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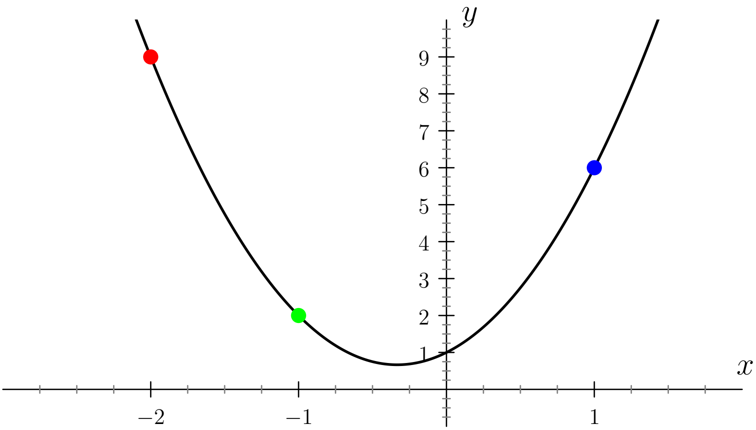

$$y = 3x^2 + 2x + 1$$

Systems of linear equations: notation

A linear equation is an expression

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n = \beta,$$

where

- $\alpha_1, \ldots, \alpha_n, \beta$ are real numbers
- $x_1, \ldots, x_n$ are variables

A system of linear equations is a sequence of one or more linear equations.

An n-tuple $(\varepsilon_1, \ldots, \varepsilon_n)$ of real numbers is a solution to the system if substituting $x_1 := \varepsilon_1, \ldots, x_n := \varepsilon_n$ to each linear equation gives a true statement.

The set of solutions is a set containing all n-tuples that are solutions.

Notation example

System of equations

$$\begin{aligned} 4a - 2b + c &= 9 \\ a - b + c &= 2 \\ a + b + c &= 6 \end{aligned}$$

with variables a, b, c.

- $(3, 2, 1)$ is a solution

$$\begin{aligned} 4 \cdot 3 - 2 \cdot 2 + 1 &= 9 \\ 3 - 2 + 1 &= 2 \\ 3 + 2 + 1 &= 6 \end{aligned}$$

- $(1, 1, 7)$ is not a solution, since

$$1 - 1 + 7 \neq 2$$

Systems of linear equations: number of solutions

- one solution, or
- no solution, or
- infinitely many solutions

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Set of solutions: $\{(3, 2, 1)\}$

Systems of linear equations: number of solutions

- one solution, or
- no solution, or
- infinitely many solutions

$$\begin{aligned} a + b \phantom{{}+c} &= 1 \\ b + c &= 1 \\ a + 2b + c &= 3 \end{aligned}$$

In any solution to first two equations:

$$a + 2b + c = (a + b) + (b + c) = 2,$$

which is incompatible with the third equation.

Set of solutions: $\emptyset$

Systems of linear equations: number of solutions

- one solution, or
- no solution, or
- infinitely many solutions

$$\begin{aligned} a + b \qquad &= 1 \\ b + c &= 1 \end{aligned}$$

For any real t, $(t, 1-t, t)$ is a solution:

$$\begin{array}{ccccccc} t & + & (1-t) & & & = & 1 \\ & & (1-t) & + & t & = & 1 \end{array}$$

Set of solutions: $\{(t, 1-t, t) : t \in \mathbf{R}\}$.

Operations preserving set of solutions

Theorem

Suppose S_1 is a system of equations and let S_2 be obtained from S_1 by the following operations

- *adding one equation to another,*
- *multiplying an equation by a non-zero real number,*
- *swapping two equations,*

or their combinations, including

- *substracting an equation from another, or*
- *adding a multiple of an equation to another.*

Then S_1 and S_2 have the same sets of solutions.

Adding one equation to another

S_1:

$$\begin{aligned}
\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n &= \alpha \\
\beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_n x_n &= \beta \\
\gamma_1 x_1 + \gamma_2 x_2 + \ldots + \gamma_n x_n &= \gamma \\
&\ldots
\end{aligned}$$

S_2:

$$\begin{array}{ccccccccc}
\alpha_1 x_1 & + & \alpha_2 x_2 & + & \ldots & + & \alpha_n x_n & = & \alpha \\
\beta_1 x_1 & + & \beta_2 x_2 & + & \ldots & + & \beta_n x_n & = & \beta \\
(\gamma_1 + \alpha_1) x_1 & + & (\gamma_2 + \alpha_2) x_2 & + & \ldots & + & (\gamma_n + \alpha_n) x_n & = & \gamma + \alpha \\
 & & & & \ldots & & & &
\end{array}$$

Adding one equation to another

Example:

S_1:

$$\begin{array}{ccccccc} x_1 & + & x_2 & + & x_3 & = & 1 \\ x_1 & + & 2x_2 & + & 3x_3 & = & 2 \\ x_1 & - & x_2 & + & 2x_3 & = & 6 \end{array}$$

S_2:

$$\begin{array}{ccccccc} x_1 & + & x_2 & + & x_3 & = & 1 \\ x_1 & + & 2x_2 & + & 3x_3 & = & 2 \\ 2x_1 & & & + & 3x_3 & = & 7 \end{array}$$

Adding one equation to another

We want: every solution to S_2 is a solution to S_1, and vice versa.

If $(e_1, \ldots, e_n)$ is a solution to S_2, then

$$\begin{array}{rcl} \alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_n e_n & = & \alpha \\ (\gamma_1 + \alpha_1) e_1 + (\gamma_2 + \alpha_2) e_2 + \ldots + (\gamma_n + \alpha_n) e_n & = & \gamma + \alpha. \end{array}$$

Hence,

$$\begin{array}{rcl} \gamma_1 e_1 + \gamma_2 e_2 + \ldots + \gamma_n e_n & = & \\ {[(\gamma_1 + \alpha_1) e_1 + (\gamma_2 + \alpha_2) e_2 + \ldots + (\gamma_n + \alpha_n) e_n] -} & & \\ {[\alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_n e_n]} & = & \\ (\gamma + \alpha) - \alpha & = & \gamma, \end{array}$$

and thus $(e_1, \ldots, e_n)$ is a solution to S_1 as well.

Multiplying by non-zero number

S_1:

$$\begin{array}{rcll} \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n & = & \alpha & \\ \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_n x_n & = & \beta & (\times k) \\ \gamma_1 x_1 + \gamma_2 x_2 + \ldots + \gamma_n x_n & = & \gamma & \\ & \ldots & & \end{array}$$

S_2:

$$\begin{array}{rcccccccc} \alpha_1 x_1 & + & \alpha_2 x_2 & + & \ldots & + & \alpha_n x_n & = & \alpha \\ (k\beta_1) x_1 & + & (k\beta_2) x_2 & + & \ldots & + & (k\beta_n) x_n & = & k\beta \\ \gamma_1 x_1 & + & \gamma_2 x_2 & + & \ldots & + & \gamma_n x_n & = & \gamma \\ & & & & \ldots & & & & \end{array}$$

Multiplying by non-zero number

Example:

S_1:

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & = & 1 \\ x_1 & + & 2x_2 & + & 3x_3 & = & 2 \qquad (\times 3) \\ x_1 & - & x_2 & + & 2x_3 & = & 6 \end{array}$$

S_2:

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & = & 1 \\ 3x_1 & + & 6x_2 & + & 9x_3 & = & 6 \\ x_1 & - & x_2 & + & 2x_3 & = & 6 \end{array}$$

Swapping two equations

S_1:

$$\begin{aligned} \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n &= \alpha \\ \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_n x_n &= \beta \\ \gamma_1 x_1 + \gamma_2 x_2 + \ldots + \gamma_n x_n &= \gamma \\ &\ldots \end{aligned}$$

S_2:

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S_2:

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & + & 2x_3 & = & 6 \\ x_1 & + & 2x_2 & + & 3x_3 & = & 2 \\ x_1 & + & x_2 & + & x_3 & = & 1 \end{array}$$

Combinations: adding a multiple of an equation

Add $k\times$ the first equation to the third one:

$$\begin{array}{ccccccc}
\alpha_1 x_1 & + & \ldots & + & \alpha_n x_n & = & \alpha \\
\beta_1 x_1 & + & \ldots & + & \beta_n x_n & = & \beta \\
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\end{array}$$

Subtracting an equation $\equiv$ adding $(-1\times)$ the equation

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Gaussian elimination: example

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 5 \\ x_1 + x_2 + x_3 + 3x_4 &= 6 \\ x_1 + 2x_2 + x_3 &= 4 \\ x_1 + 3x_2 + x_3 + x_4 &= 6 \end{aligned}$$

Swap equations so that the second has non-zero coefficient at x_2:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 5 \\ x_2 - x_4 - x_5 &= -1 \\ 2x_4 - x_5 &= 1 \\ 2x_2 - x_5 &= 1 \end{aligned}$$

Eliminate x_1 by subtracting the first equation from others:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 5 \\ 2x_4 - x_5 &= 1 \\ x_2 - x_4 - x_5 &= -1 \\ 2x_2 - x_5 &= 1 \end{aligned}$$

Eliminate x_2 by subtracting $2\times$ the second equation from 4th:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 5 \\ x_2 - x_4 - x_5 &= -1 \\ 2x_4 - x_5 &= 1 \\ 2x_4 + x_5 &= 3 \end{aligned}$$

Gaussian elimination: example

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Gaussian elimination: example

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Gaussian elimination: example continued

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 5 \\ x_1 + x_2 + x_3 + 3x_4 \phantom{{}+x_5} &= 6 \\ x_1 + 2x_2 + x_3 \phantom{{}+3x_4+x_5} &= 4 \\ x_1 + 3x_2 + x_3 + x_4 \phantom{{}+x_5} &= 6 \end{aligned}$$

After eliminating x_1 and x_2:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 5 \\ x_2 \phantom{{}+x_3} - x_4 - x_5 &= -1 \\ 2x_4 - x_5 &= 1 \\ 2x_4 + x_5 &= 3 \end{aligned}$$

Eliminate x_4 by subtracting the 3rd equation from the 4th:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 5 \\ x_2 \phantom{{}+x_3} - x_4 - x_5 &= -1 \\ 2x_4 - x_5 &= 1 \\ 2x_5 &= 2 \end{aligned}$$

Gaussian elimination: example continued

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 5 \\ x_1 + x_2 + x_3 + 3x_4 &= 6 \\ x_1 + 2x_2 + x_3 &= 4 \\ x_1 + 3x_2 + x_3 + x_4 &= 6 \end{aligned}$$

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Gaussian elimination: example solution

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 5 \\ x_1 + x_2 + x_3 + 3x_4 &= 6 \\ x_1 + 2x_2 + x_3 &= 4 \\ x_1 + 3x_2 + x_3 + x_4 &= 6 \end{aligned}$$

After Gaussian elimination:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 5 \\ x_2 - x_4 - x_5 &= -1 \\ 2x_4 - x_5 &= 1 \\ 2x_5 &= 2 \end{aligned}$$

Backward substitution:

4th equation: $x_5 = 1$

3rd equation:

$x_4 = (1 + x_5)/2 = 1$

2nd equation:

- x_3 can be arbitrary; $x_3 = t$ for any $t \in \mathbf{R}$
- $x_2 = -1 + x_4 + x_5 = 1$

1st equation:

$x_1 = 5 - x_2 - x_3 - x_4 - x_5 = 2 - t$

Set of solutions:

$$\{(2 - t, 1, t, 1, 1) : t \in \mathbf{R}\}$$

Gaussian elimination: example solution

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$$\begin{aligned}
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x_1 + x_2 + x_3 + 3x_4 \phantom{{}+x_5} &= 6 \\
x_1 + 2x_2 + x_3 \phantom{{}+3x_4+x_5} &= 4 \\
x_1 + 3x_2 + x_3 + x_4 \phantom{{}+x_5} &= 6
\end{aligned}$$

After Gaussian elimination:

$$\begin{aligned}
x_1 + x_2 + x_3 + x_4 + x_5 &= 5 \\
x_2 \phantom{{}+x_3} - x_4 - x_5 &= -1 \\
2x_4 - x_5 &= 1 \\
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$x_1 = 5 - x_2 - x_3 - x_4 - x_5 = 2 - t$

Set of solutions:

$$\{(2 - t, 1, t, 1, 1) : t \in \mathbf{R}\}$$

Matrix notation

Instead of

$$\begin{array}{rcl} \alpha_{1,1}x_1 + \alpha_{1,2}x_2 + \ldots + \alpha_{1,n}x_n & = & \beta_1 \\ \alpha_{2,1}x_1 + \alpha_{2,2}x_2 + \ldots + \alpha_{2,n}x_n & = & \beta_2 \\ & \ldots & \\ \alpha_{m,1}x_1 + \alpha_{m,2}x_2 + \ldots + \alpha_{m,n}x_n & = & \beta_m, \end{array}$$

we write

$$\begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \ldots & \alpha_{1,n} \\ \alpha_{2,1} & \alpha_{2,2} & \ldots & \alpha_{2,n} \\ & \ldots & & \\ \alpha_{m,1} & \alpha_{m,2} & \ldots & \alpha_{m,n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \ldots \\ \beta_m \end{pmatrix}$$

Matrix notation: example

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 5 \\ x_1 + x_2 + x_3 + 3x_4 \phantom{{}+x_5} &= 6 \\ x_1 + 2x_2 + x_3 \phantom{{}+3x_4+x_5} &= 4 \\ x_1 + 3x_2 + x_3 + x_4 \phantom{{}+x_5} &= 6 \end{aligned}$$

is the same as

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 3 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 4 \\ 5 \end{pmatrix}$$

Matrix notation

$$A = \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \dots & \alpha_{1,n} \\ \alpha_{2,1} & \alpha_{2,2} & \dots & \alpha_{2,n} \\ & & \dots & \\ \alpha_{m,1} & \alpha_{m,2} & \dots & \alpha_{m,n} \end{pmatrix},$$

where $\alpha_{1,1}, \dots, \alpha_{m,n}$ are real numbers, is an $m \times n$ matrix

- $m =$ number of rows, $n =$ number of columns. Matrix is square if $m = n$.
- $A_{i,j}$ denotes the element ($\alpha_{i,j}$) in the i-th row and j-th column.
- $A_{i,\star} = (\alpha_{i,1}, \alpha_{i,2}, \dots, \alpha_{i,n})$ denotes the i-th row of A.
- $A_{\star,j} = \begin{pmatrix} \alpha_{1,j} \\ \alpha_{2,j} \\ \dots \\ \alpha_{m,j} \end{pmatrix}$ denotes the j-th column of A.

Example

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}$$

is a 3×4 matrix.

- 3 rows, 4 columns
- $A_{2,3} = 7$
- the second row: $A_{2,\star} = (5, 6, 7, 8)$
- the third column: $A_{\star,3} = \begin{pmatrix} 3 \\ 7 \\ 11 \end{pmatrix}$

More notation

From now on, we will (generally) use

- uppercase letters A, B, ... for matrices
- lowercase letters a, b, x, y, ... for matrices with one column (~vectors)
- lowercase letters m, n, p, ... for integers
- greek alphabet letters α, β, ... and lowercase letters s, t, ... for real numbers

More matrix notation

For matrices $A = \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \dots & \alpha_{1,n} \\ & & \dots & \\ \alpha_{m,1} & \alpha_{m,2} & \dots & \alpha_{m,n} \end{pmatrix}$ and $B = \begin{pmatrix} \beta_{1,1} & \beta_{1,2} & \dots & \beta_{1,p} \\ & & \dots & \\ \beta_{m,1} & \beta_{m,2} & \dots & \beta_{m,p} \end{pmatrix}$ with the same number of rows, let

$$(A|B) = \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \dots & \alpha_{1,n} & \beta_{1,1} & \beta_{1,2} & \dots & \beta_{1,p} \\ & & & & \dots & & & \\ \alpha_{m,1} & \alpha_{m,2} & \dots & \alpha_{m,n} & \beta_{m,1} & \beta_{m,2} & \dots & \beta_{m,p} \end{pmatrix}$$

be the $m \times (n+p)$ matrix obtained by putting B to the right of A.

Even more matrix notation

For system of equations $Ax = b$,

- A is the matrix of the system
- $(A|b)$ is the extended matrix of the system

Example: System

$$\begin{aligned} 4x_1 - 2x_2 + x_3 &= 9 \\ x_1 - x_2 + x_3 &= 2 \\ x_1 + x_2 + x_3 &= 6 \end{aligned}$$

has

- matrix $\begin{pmatrix} 4 & -2 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$
- extended matrix $\left(\begin{array}{ccc|c} 4 & -2 & 1 & 9 \\ 1 & -1 & 1 & 2 \\ 1 & 1 & 1 & 6 \end{array}\right)$

Gaussian elimination on matrices

We can

- add a row to another
- multiply a row by a non-zero real number
- swap rows
- subtract a row from another
- add a multiple of a row to another

We call these operations elementary row operations. Two matrices A and B are row-equivalent (we write $A \sim B$) if B can be obtained from A by a sequence of elementary row operations.

Observation

If $A \sim B$, then $B \sim A$. That is, elementary row operations are invertible and A can also be obtained from B by a sequence of elementary row operations.

Gaussian elimination on matrices: example

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 5 \\ x_1 + x_2 + x_3 + 3x_4 &= 6 \\ x_1 + 2x_2 + x_3 &= 4 \\ x_1 + 3x_2 + x_3 + x_4 &= 6 \end{aligned}$$

extended matrix:

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 5 \\ 1 & 1 & 1 & 3 & 0 & 6 \\ 1 & 2 & 1 & 0 & 0 & 4 \\ 1 & 3 & 1 & 1 & 0 & 6 \end{array}\right) \sim$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 5 \\ 0 & 0 & 0 & 2 & -1 & 1 \\ 0 & 1 & 0 & -1 & -1 & -1 \\ 0 & 2 & 0 & 0 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 5 \\ 0 & 1 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 2 & -1 & 1 \\ 0 & 2 & 0 & 0 & -1 & 1 \end{array}\right) \sim$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 5 \\ 0 & 1 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 2 & -1 & 1 \\ 0 & 0 & 0 & 2 & 1 & 3 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 5 \\ 0 & 1 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 & 2 & 2 \end{array}\right)$$

Gaussian elimination on matrices: example

$$
\begin{aligned}
x_1 + x_2 + x_3 + x_4 + x_5 &= 5 \\
x_1 + x_2 + x_3 + 3x_4 \phantom{{}+x_5} &= 6 \\
x_1 + 2x_2 + x_3 \phantom{{}+3x_4+x_5} &= 4 \\
x_1 + 3x_2 + x_3 + x_4 \phantom{{}+x_5} &= 6
\end{aligned}
$$

extended matrix:

$$
\left(\begin{array}{ccccc|c}
1 & 1 & 1 & 1 & 1 & 5 \\
1 & 1 & 1 & 3 & 0 & 6 \\
1 & 2 & 1 & 0 & 0 & 4 \\
1 & 3 & 1 & 1 & 0 & 6
\end{array}\right) \sim
$$

$$
\left(\begin{array}{ccccc|c}
1 & 1 & 1 & 1 & 1 & 5 \\
0 & 1 & 0 & -1 & -1 & -1 \\
0 & 0 & 0 & 2 & -1 & 1 \\
0 & 0 & 0 & 0 & 2 & 2
\end{array}\right) \rightarrow
$$

$$
\begin{aligned}
x_1 + x_2 + x_3 + x_4 + x_5 &= 5 \\
x_2 \phantom{{}+x_3} - x_4 - x_5 &= -1 \\
2x_4 - x_5 &= 1 \\
2x_5 &= 2
\end{aligned}
$$

Definition

Let A be an $m \times n$ matrix. For $1 \leq i \leq m$, let $p_i = \min\{j : A_{i,j} \neq \emptyset\}$ denote the index of the first non-zero element in the i-th row. We say that A is in Row Echelon Form (REF) if for some $r \leq m$,

- each of first r rows of A contains a non-zero element,
- the rows $r+1, \dots, m$ are zero, and
- $p_1 < p_2 < \dots < p_r$.

Integers $p_1, \dots, p_r$ are called basis column indices.

Example:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 5 \\ 0 & 1 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$p_1 = 1$$
$$p_2 = 2$$
$$p_3 = 4$$
$$p_4 = 5$$

Gaussian elimination: formal description

For an $m \times n$ matrix A:

1. $r := 1$, $c := 1$
2. If $A_{i,j} = 0$ for all $i \geq r$ and $j \geq c$, then end.
3. Let $c := \min\{j \geq c : A_{i,j} \neq 0 \text{ for some } i \geq r\}$.
    - Find first column after current position with non-zero entry in row $\geq r$.
4. Choose arbitrary $i \geq r$ such that $A_{i,c} \neq 0$, and swap i-th and r-th row.
    - So now $A_{r,c} \neq \emptyset$.
5. For every $i > r$, subtract $\frac{A_{i,c}}{A_{r,c}}$-times the r-th row from the i-th row.
    - So that all entries in the column below $A_{r,c}$ are zero.
6. Let $r := r + 1$, $c := c + 1$ and repeat from step 2.

Properties of Row Echelon Form

Theorem

Gaussian elimination applied to matrix B returns a row-equivalent matrix A in REF.

There may exist many different matrices in REF that are row-equivalent to B. However:

Theorem (for now without proof)

If A and A' are any matrices in REF and $A \sim A'$, then A and A' have the same basis column indices. In particular, they have the same number of non-zero rows.

This motivates the following definition.

Definition

The rank of a matrix B (denoted by $\mathrm{rank}(B)$) is the number of non-zero rows of a row-equivalent matrix in REF.

Rank: example

Problem

Determine the rank of $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 2 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

$$A \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The last matrix is in REF and has 2 non-zero rows, hence

$$\mathrm{rank}(A) = 2.$$

Gaussian elimination: determining the set of solutions

Consider a system of linear equations $Ax = b$ with m equations and n variables $x_1, \ldots, x_n$.

- Let $(A'|b')$ be the result of Gaussian elimination of $(A|b)$.
  - $(A'|b')$ is in REF, with basis column indices $p_1 < \ldots < p_r$.

Gaussian elimination: determining the set of solutions

Consider a system of linear equations $Ax = b$ with m equations and n variables $x_1, \ldots, x_n$.

- Let $(A'|b')$ be the result of Gaussian elimination of $(A|b)$.
  - $(A'|b')$ is in REF, with basis column indices $p_1 < \ldots < p_r$.

If $p_r = n + 1$, then the system has no solution. Example:

$$\begin{aligned} x_1 + x_2 \phantom{{}+x_3} &= 1 \\ x_2 + x_3 &= 1 \\ x_1 + 2x_2 + x_3 &= 3 \end{aligned} \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 1 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right) \rightarrow$$

$$\begin{aligned} x_1 + x_2 \phantom{{}+0x_3} &= 1 \\ x_2 + x_3 &= 1 \\ 0x_1 + 0x_2 + 0x_3 &= 1 \end{aligned}$$

The last equation cannot be satisfied.

Gaussian elimination: determining the set of solutions

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  - $(A'|b')$ is in REF, with basis column indices $p_1 < \ldots < p_r$.

If $r = n$ and $p_1 = 1$, $p_2 = 2, \ldots, p_n = n$, then the system has one solution, obtained by backward substitution.

$$
\begin{aligned}
x_n &= \frac{b'_n}{A'_{n,n}} \\
x_{n-1} &= \frac{b'_{n-1} - A'_{n-1,n} x_n}{A'_{n-1,n-1}} \\
x_{n-2} &= \frac{b'_{n-2} - A'_{n-2,n-1} x_{n-1} - A'_{n-2,n} x_n}{A'_{n-2,n-2}} \\
&\ldots \\
x_1 &= \frac{b'_1 - A'_{1,2} x_2 - A'_{1,3} x_3 - \ldots - A'_{1,n} x_n}{A'_{1,1}}
\end{aligned}
$$

Gaussian elimination: determining the set of solutions

Consider a system of linear equations $Ax = b$ with m equations and n variables $x_1, \ldots, x_n$.

- Let $(A'|b')$ be the result of Gaussian elimination of $(A|b)$.
  - $(A'|b')$ is in REF, with basis column indices $p_1 < \ldots < p_r$.

If $r = n$ and $p_1 = 1$, $p_2 = 2$, $\ldots$, $p_n = n$, then the system has one solution, obtained by backward substitution.

Example:

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_2 + x_3 &= 1 \\ x_1 + x_2 + x_3 &= 3 \end{aligned} \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right) \rightarrow$$

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_2 + x_3 &= 1 \\ x_3 &= 2 \end{aligned} \rightarrow \begin{aligned} x_3 &= 2 \\ x_2 &= 1 - x_3 = -1 \\ x_1 &= 1 - x_2 = 2 \end{aligned}$$

Gaussian elimination: determining the set of solutions

Consider a system of linear equations $Ax = b$ with m equations and n variables $x_1, \ldots, x_n$.

- Let $(A'|b')$ be the result of Gaussian elimination of $(A|b)$.
  - $(A'|b')$ is in REF, with basis column indices $p_1 < \ldots < p_r$.

If $r < n$ and $p_r \leq n$, then the system has infinitely many solutions. The values of basis column variables $x_{p_1}, \ldots x_{p_r}$ can be obtained by backward substitution, with the other variables acting as parameters. Example:

$$\begin{aligned} x_1 + \quad x_2 \qquad\quad + \quad x_4 &= 1 \\ x_2 + x_3 + \quad x_4 &= 1 \\ x_1 + 2x_2 + x_3 + 2x_4 &= 2 \end{aligned} \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 & 2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\rightarrow \begin{aligned} x_1 + x_2 \qquad + x_4 &= 1 \\ x_2 + x_3 + x_4 &= 1 \end{aligned} \rightarrow \begin{aligned} x_4 &= \alpha \\ x_3 &= \beta \\ x_2 &= 1 - x_3 - x_4 = 1 - \alpha - \beta \\ x_1 &= 1 - x_2 - x_4 = \beta \end{aligned}$$

for any $\alpha, \beta \in \mathbf{R}$.

Gaussian elimination: determining the set of solutions

Consider a system of linear equations $Ax = b$ with m equations and n variables $x_1, \ldots, x_n$.

- Let $(A'|b')$ be the result of Gaussian elimination of $(A|b)$.
  - $(A'|b')$ is in REF, with basis column indices $p_1 < \ldots < p_r$.

Summary:

- If $p_r = n + 1$, then the system has no solution.
- If $r = p_r = n$, then the system has one solution.
- Otherwise, the system has infinitely many solutions.

Note that $A \sim A'$ and A' is in REF. If $p_r = n+1$, then A' has $r - 1$ non-zero rows, otherwise A' has r non-zero rows.

Theorem

The system $Ax = b$ has no solution if and only of $rank(A|b) > rank(A)$. If $rank(A|b) = rank(A) = n$, then the system has one solution, while if $rank(A|b) = rank(A) < n$, then the system has infinitely many solutions.