List coloring, Nullstellensatz, polynomial method

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1 Chevalley-Warning theorem and regular subgraphs

Theorem 1 (Chevalley-Warning theorem)**.** *Suppose p is a prime and f_1, …, f_r are polynomials over Z_p in n variables and of total degrees d_1, …, and d_r. If $\sum_{i=1}^r d_i < n$, then the number of solutions $f_1(\vec{x}) = 0$, …, $f_r(\vec{x}) = 0$ is divisible by p.*

We did not show the proof (it can be found in the Czech version of the notes).

Theorem 2. *Let G be a multigraph of minimum degree at least four and maximum degree exactly five (i.e., G is not 4-regular). Then G contains a 3-regular submultigraph.*

Proof. Let $r = |V(G)|$. For $v \in V(G)$, let us define

$$f_v = \sum_{e \text{ incident with } v} x_e^2.$$

as a polynomial over Z_3. The number of variables used in all the polynomials is $|E(G)| = \frac{1}{2}\sum_{v\in V(G)} \deg v > 2r = \sum_{v\in V(G)} \deg(f_v)$. Theorem 1 implies that the number of the solutions to the system $f_v(\vec{x}) = 0$ for $v \in V(G)$ is divisible 3. The system has a trivial all-zeros solution, and thus it has also (at least two) non-zero solutions. Let $X = \{e \in E(G) : x_e \neq 0\}$ for a non-zero solution. Since $x^2 = 1$ for every $x \in Z_3 \setminus \{0\}$ and G has maximum degree five, $f_v(\vec{x}) = 0$ is equivalent to the claim that the degree of v in the subgraph $(V(G), X)$ is 0 or 3. $\square$

The assumption that G is not 4-regular is necessary, since the triangle with doubled edges has no 3-regular subgraph.

2 List coloring

A *list assignment* for a graph G is a function L assigning a set of colors (a *list*) to each vertex of G. An *L-coloring* of a graph G is a proper coloring φ of G such that $\varphi(v) \in L(v)$ for every $v \in V(G)$. The *list chromatic number* $\chi_l(G)$ of a graph G is the smallest integer k such that G is L-colorable for every assignment L of lists of size at least k to vertices of G.

Observation 3.

$$\begin{aligned} \chi_l(G) &\geq \chi(G) \\ \chi_l(G) &\leq d+1 \textit{ if } G \textit{ is } d\textit{-degenerate} \\ \chi_l(C_n) &= \chi(C_n) \end{aligned}$$

Lemma 4.

$$\chi_l(K_{n,n^n}) > n.$$

Proof. Let the vertices of the graph be $v_1, \ldots, v_n$, and $w_{i_1,\ldots,i_n}$ for $i_1, \ldots, i_n \in \{1, \ldots, n\}$. Let $L(v_k) = \{c_{k,1}, \ldots, c_{k,n}\}$. The vertices $w_\star$ are assigned all n-element lists which intersect each of $L(v_1), \ldots, L(v_n)$ in exactly one element; i.e., $L(w_{i_1,\ldots,i_n}) = \{c_{1,i_1}, c_{2,i_2}, \ldots, c_{n,i_n}\}$. Consider any L-coloring of $v_1, \ldots, v_n$, by some colors barvami $c_{1,i_1}, \ldots, c_{n,i_n}$. Then it is not possible to color the vertex $w_{i_1,\ldots,i_n}$ from its lists, as all the colors from $L(w_{i_1,\ldots,i_n})$ are already used on $v_1, \ldots, v_n$. Hence, K_{n,n^n} is not L-colorable. □

3 List-colorablity of planar graphs

Every planar graph has list chromatic number at most 5, and there are planar graphs of list chromatic number exactly five. We did not do detailed arguments in the class (though they are shown in the Czech version of the notes). The graph $K_{2,4}$ is planar, bipartite, and by Lemma 4 has list chromatic number three. As another example of the polynomial method, we are going to show that every planar bipartite graph has list chromatic number at most three.

4 Nullstellensatz

We need the following algebraic statement, which is easy to prove by induction on the number of variables.

Lemma 5. *Let $p(x_1, \dots, x_n)$ be a polynomial in n variables and for $i \in \{1, \dots, n\}$, let d_i be the maximum degree of the variable x_i in p. For $i \in \{1, \dots, n\}$, let S_i be a set of more than d_i complex numbers. If $p \neq 0$, then there exist $s_1 \in S_1, \dots, s_n \in S_n$ such that $p(s_1, \dots, s_n) \neq 0$.*

Let G be a graph with vertices $\{v_1, \dots, v_n\}$ and let $\vec{G}$ be any orientation of G. The *graph polynomial* $P_{\vec{G}}$ is defined as

$$P_{\vec{G}}(x_1, \dots, x_n) = \prod_{(v_i, v_j) \in E(\vec{G})} (x_j - x_i).$$

Note that $P_{\vec{G}}(c_1, \dots, c_n) \neq 0$ if and only if $c_1, \dots, c_n$ is a proper coloring of the vertices of G.

Theorem 6. *Let G be a graph with vertices $\{v_1, \dots, v_n\}$ and let $\vec{G}$ be an orientation of G. Let $d_1, \dots, d_n$ be integers and L a list assignment for G such that $|L(v_i)| > d_i$ for $1 \le i \le n$. If the coefficient at $x_1^{d_1} \dots x_n^{d_n}$ in $P_{\vec{G}}$ is non-zero, then G is L-colorable.*

Proof. Without loss of generality, we can assume that $|L(v_i)| = d_i + 1$ and the elements of $L(v_i)$ are complex numbers. Let $p_i(x) = \prod_{c \in L(v_i)} (x - c)$ for $i = 1, \dots, n$. Then $p_i(c) = 0$ for every $c \in L(v_i)$. Let $q_i = x^{d_i+1} - p_i$; then q_i is a polynomial of degree at most d_i and $q_i(c) = c^{d_i+1}$ for every $c \in L(v_i)$. Let P be the polynomial obtained from $P_{\vec{G}}$ by repeatedly substituting q_i for x^{d_i+1}, for each $i \in \{1, \dots, n\}$. Then $P(c_1, \dots, c_n) = P_{\vec{G}}(c_1, \dots, c_n)$ for every $c_1 \in L(v_1), \dots, c_n \in L(v_n)$, and for each i, the degree of x_i in P is at most d_i. Moreover, the coefficient at $x_1^{d_1} \dots x_n^{d_n}$ is the same in P and in $P_{\vec{G}}$, since every monomial of $P_{\vec{G}}$ has the same total degree (equal to $|E(G)|$) and the substitutions only result in monomials of smaller total degree. It follows that $P \neq 0$, and Lemma 5 implies that there exist $c_1 \in L(v_1), \dots,$ and $c_n \in L(v_n)$ such that $P_{\vec{G}}(c_1, \dots, c_n) = P(c_1, \dots, c_n) \neq 0$. Then $c_1, \dots, c_n$ is a proper L-coloring of G. $\square$

Let $\vec{G}$ be a fixed orientation of G. If $\vec{G}'$ is another orientation of G differing from $\vec{G}$ on exactly p edges, then let $\operatorname{sgn}(\vec{G}') = (-1)^p$.

Observation 7. *Let G be a graph with vertices $\{v_1, \dots, v_n\}$ and let $\vec{G}$ be an orientation of G. Let $\mathcal{O}_{d_1,\dots,d_n}$ be the set of all orientations of G in which v_i has indegree d_i for $i = 1, \dots, n$. The absolute value of the coefficient at the monomial $x_1^{d_1} \dots x_n^{d_n}$ in $P_{\vec{G}}$ is*

$$\left| \sum_{\vec{G}' \in \mathcal{O}_{d_1,\dots,d_n}} \operatorname{sgn}(\vec{G}') \right|.$$

Any two orientations with the same indegrees differ only by reversing the edges of an Eulerian subgraph. Hence:

Corollary 8. *Let G be a graph with vertices $\{v_1, \ldots, v_n\}$, let $\vec{G}$ be an orientation of G, and for $1 \leq i \leq n$, let d_i be the indegree of v_i. Let $\mathcal{E}$ consist of all subsets of edges of $\vec{G}$ forming an Eulerian subgraph. The absolute value of the coefficient at the monomial $x_1^{d_1} \ldots x_n^{d_n}$ in $P_{\vec{G}}$ is equal to*

$$\left|\sum_{X \in \mathcal{E}} (-1)^{|X|}\right|.$$

If G is bipartite, then every Eulerian subgraph of an orientation of G has even number of edges (and there is at least one, the empty one). Combining these results, we get the following claim.

Corollary 9. *Let G be a bipartite graph with vertices $\{v_1, \ldots, v_n\}$, let $\vec{G}$ be an orientation of G, and for $1 \leq i \leq n$, let d_i be the indegree of v_i. Then G is L-colorable for every list assignment L such hat $|L(v_i)| > d_i$.*

Hall's theorem implies that every bipartite planar graph has an orientation with maximum indegree at most two, and thus every bipartite planar graph has list chromatic number at most three.