Fractional coloring

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1 Introduction and definitions

Note that a graph is k-colorable if and only if it can be covered by at most k independent sets; i.e., we can assign value 0 or 1 to each independent set so that the sum of the assigned values is at most k and each vertex is contained in an independent set having value 1. This reformulation motivates the following fractional relaxation. Let $\mathcal{I}(G)$ denote the set of all independent sets in G.

Definition 1. *The* fractional chromatic number $\chi_f(G)$ *of a graph G is the minimum of*

$$\sum_{I\in\mathcal{I}(G)} x_I,$$

over all $x_I \ge 0$ for $I \in \mathcal{I}(G)$ such that

$$\sum_{I\in\mathcal{I}(G),v\in I} x_I \ge 1$$

holds for all $v \in V(G)$.

By LP duality, we have the following alternate formulation.

Observation 1. *The fractional chromatic number of a graph G is the maximum of*

$$\sum_{v\in V(G)} y_v,$$

over all $y_v \ge 0$ for $v \in V(G)$ such that

$$\sum_{v\in I} y_v \le 1$$

holds for all $I \in \mathcal{I}(G)$.

Let $w : V(G) \to \mathbf{R}_0^+$ be an assignment of nonnegative weights to vertices. For $X \subseteq V(G)$, let us define $w(X) = \sum_{v \in X} w(v)$. Let

$$\alpha_w(G) = \max\{w(I) : I \in \mathcal{I}(G)\}.$$

Lemma 2.

$$\chi_f(G) = \max \frac{w(V(G))}{\alpha_w(G)} \text{ over all } w : V(G) \to \mathbf{R}_0^+, \text{ not identically } 0.$$

Proof. Consider any $w : V(G) \to \mathbf{R}_0^+$, not identically 0. Note that $\alpha_w(G) \geq \max\{w(v) : v \in V(G)\} > 0$. Let $y_v = \frac{w(v)}{\alpha_w(G)}$ for all $w \in V(G)$. This assignment satisfies the constraints of Observation 1, and thus $\chi_f(G) \geq \sum_{v \in V(G)} y_v = \frac{w(V(G))}{\alpha_w(G)}$.

Conversely, let y_v be the assignment satisfying the constraints of Observation 1 such that $\sum_{v \in V(G)} y_v = \chi_f(G)$. Let $w(v) = y_v$; the constraints imply $\alpha_w(G) \leq 1$, and thus for this weight assignment w, we have $\chi_f(G) = w(V(G)) \leq \frac{w(V(G))}{\alpha_w(G)}$, which together with the previous paragraph implies $\chi_f(G) = \frac{w(V(G))}{\alpha_w(G)}$. □

Corollary 3. *If G is vertex transitive, then $\chi_f(G) = \frac{|V(G)|}{\alpha(G)}$.*

Proof. Considering the weight assignment w such that $w(v) = 1$ for all $v \in V(G)$, Lemma 2 implies $\chi_f(G) \geq \frac{|V(G)|}{\alpha(G)}$. Let k be the number of independent sets of size $\alpha(G)$ that contain a vertex v of G (since G is vertex transitive, this number is independent on the choice of v). Let $\mathcal{I}_{\max}(G)$ be the set of all independent sets of G of size $\alpha(G)$. For $I \in \mathcal{I}(G)$, let $x_I = 1/k$ if $|I| = \alpha(G)$ and $x_I = 0$ otherwise. Clearly the assignment x_I satisfies the constraints from Definition 1, and thus

$$\begin{aligned}
\chi_f(G) &\leq \sum_{I \in \mathcal{I}(G)} x_I = \frac{1}{k}|\mathcal{I}_{\max}(G)| \\
&= \frac{1}{k\alpha(G)} \sum_{I \in \mathcal{I}_{\max}(G)} |I| \\
&= \frac{1}{k\alpha(G)} |\{(I, v) : I \in \mathcal{I}_{\max}(G), v \in I\}| \\
&= \frac{k|V(G)|}{k\alpha(G)} = \frac{|V(G)|}{\alpha(G)}.
\end{aligned}$$

Hence, $\chi_f(G) = \frac{|V(G)|}{\alpha(G)}$. □

A *set coloring* of G is a function φ that assigns a set to each vertex of G, such that $\varphi(u) \cap \varphi(v) = \emptyset$ for all $uv \in E(G)$. An *$(a:b)$-coloring* of G is a set coloring φ such that $|\varphi(v)| \geq b$ for all $v \in V(G)$ and $\left|\bigcup_{v \in V(G)} \varphi(v)\right| \leq a$.

Lemma 4.

$$\chi_f(G) = \min\{a/b : \ G \text{ has an } (a:b)\text{-coloring.}\}$$

Proof. Let φ be an $(a:b)$-coloring of G, where w.l.o.g. $\varphi(v) \subseteq [a]$ for all $v \in V(G)$. For a color $c \in [a]$, let $I_c = \{v \in V(G) : c \in \varphi(v)\}$, and for $I \in \mathcal{I}(G)$, let a_I be the number of colors $c \in [a]$ such that $I_c = I$. Let $x_I = a_I/b$. Note that for $v \in V(G)$, we have

$$\sum_{I \in \mathcal{I}(G), v \in I} x_I = \frac{|\varphi(v)|}{b} \geq 1,$$

and that

$$\sum_{I \in \mathcal{I}(G)} x_I = \frac{a}{b},$$

and thus $\chi_f(G) \leq a/b$.

Conversely, consider an optimal solution to the linear program from Definition 1. Since all the coefficients are integers, we can assume that this solution is rational; hence, there exists a positive integer b such that for all $I \in \mathcal{I}(G)$, there exists an integer a_I such that $x_I = a_I/b$. Let $A = \{(I,i) : I \in \mathcal{I}(G), i \in [a_I]\}$, and for $v \in V(G)$, let $\varphi(v) = \{(I,i) : I \in \mathcal{I}(G), v \in I, i \in [a_I]\}$. Then φ is a set coloring of G by subsets of A,

$$|A| = \sum_{I \in \mathcal{I}(G)} a_I = b \sum_{I \in \mathcal{I}(G)} x_I = b\chi_f(G),$$

and

$$|\varphi(v)| = \sum_{I \in \mathcal{I}(G), v \in I} a_I = b \sum_{I \in \mathcal{I}(G), v \in I} x_I \geq b$$

for every $v \in V(G)$. Hence, φ is a $(b\chi_f(G) : b)$-coloring of G. □

A function $f : V(G) \to V(H)$ is a *homomorphism* if $f(u)f(v) \in E(H)$ for every $uv \in E(G)$. If there exists a homomorphism from G to H, we write $G \to H$. The *Kneser graph* $K_{a:b}$ is the graph whose vertices are subsets of $[a]$ of size b and two such sets are adjacent iff they are disjoint.

Observation 5. *A graph G is k-colorable iff $G \to K_k$. A graph G has an $(a:b)$-coloring iff $G \to K_{a:b}$.*

2 Relationship to ordinary chromatic number

The *d-dimensional sphere* S_d is the boundary of ball in $d+1$ dimensions; i.e., S_0 are two points, S_1 is the circle, S_2 is the sphere, ...

Theorem 6 (Borsuk-Ulam). *Let $A_1, \ldots, A_{d+1}$ be subsets of the d-dimensional sphere S_d, each of them open or closed. If $A_1 \cup \ldots \cup A_{d+1} = S_d$, then there exists $i \in [d+1]$ and $x \in S_d$ such that both x and $-x$ belong to A_i.*

Theorem 7. *Let $a \geq 2b$ be integers. The Kneser graph $K_{a:b}$ has chromatic number $a - 2b + 2$.*

Proof. Consider a set $S \in V(K_{a:b})$. Let $\varphi(S) = \min S$ if $\min S \leq a - 2b + 1$, and $\varphi(S) = a - 2b + 2$ otherwise. This is a proper coloring: If $\varphi(S_1) = \varphi(S_2)$, then either $\min S_1 = \min S_2$ or $S_1, S_2 \subseteq \{a - 2b + 2, \ldots, a\}$. In either case, $S_1 \cap S_2 \neq \emptyset$ (in the latter case, this is because both sets of size b are subsets of a set of size $2b - 1$), and thus $S_1 S_2 \notin E(K_{a:b})$.

Suppose now for a contradiction that φ is a proper $(a - 2b + 1)$-coloring of $K_{a:b}$. Let $d = a - 2b + 1$ and let $p_1, \ldots, p_a$ be points of the d-dimensional sphere in general position (i.e., no $d+1$ of them lie on a plane passing through the center of S_d). For $c \in [a - 2b + 1]$, let $A_c \subseteq S_d$ consists of the points p of the spere such that there exists $S_{p,c} \in V(K_{a:b})$ with $\varphi(S_{p,c}) = c$ and the points $\{p_i : i \in S_{p,c}\}$ lie in the open half-sphere centered at p. Clearly, the sets $A_1, \ldots, A_{a-2b+1}$ are open. Let $A_{a-2b+2} = S_d \setminus \bigcup_{c \in [a-2b+1]} A_c$; this set is closed. By Theorem 6, there exists $c \in [a - 2b + 2]$ and a point $p \in S_d$ such that $p, -p \in A_c$.

If $c \in [a - 2b + 1]$, this means that there exist vertices $S_{p,c}$ and $S_{-p,c}$ of $K_{a:b}$ both of color c. However, the point sets in S_d that represent them are disjoint (they are contained in opposite open half-spheres), and thus $S_{p,c} S_{-p,c} \in E(K_{a:b})$, contradicting the assumption that φ is proper.

If $c = a - 2b + 2$, then note that each of the open half-spheres centered at p and at $-p$ contains at most $b - 1$ of the points $p_1, \ldots, p_a$ (as otherwise a b-tuple of them would represent a vertex S of $K_{a:b}$ and p or $-p$ would belong to $A_{\varphi(S)}$, contradicting the choice of A_{a-2b+2}). This means that the remaining at least $a - 2(b - 1) = d + 1$ points lie in the complement of these two opposite half-spheres. But then they lie on a plane passing through the center of S_d, contradicting the choice of the points in general position. □

Corollary 8. *If $a \geq 2b$ and G has an $(a : b)$-coloring, then $\chi(G) \leq a - 2b + 2$ (and this bound cannot be improved).*

Proof. By Theorem 7 and Observation 5, $G \to K_{a,b} \to K_{a-2b+2}$. The bound cannot be improved by Theorem 7, since $G = K_{a:b}$ is possible. $\square$

Corollary 9. *For every positive integer c, there exist graphs with fractional chromatic number at most $2 + 1/c$, but with arbitrarily large chromatic number.*

Proof. The graph $K_{(2c+1)b:bc}$ has fractional chromatic number $\frac{(2c+1)b}{bc} = 2+1/c$ and by Theorem 7, its chromatic number is $b + 2$, which can be arbitrarily large. $\square$

Corollary 10. *Let a and b be coprime integers such that $a \geq 2b + 1$. Let $a' > a$ and $b' > b$ be integers such that $a'/b' = a/b$. Then $K_{a',b'}$ is not $(a : b)$-colorable.*

Proof. By Theorem 7, $K_{a:b} \to K_{a-2b+2}$, while $K_{a':b'} \not\to K_{a'-2b'+1}$. Since a and b are coprime, $\gamma = a'/a = b'/b > 1$ is an integer, and thus $\gamma \geq 2$. Note that $(a' - 2b' + 1) - (a - 2b + 2) = (\gamma - 1)(a - 2b) - 1 \geq 0$, and thus $K_{a-2b+2} \to K_{a'-2b'+1}$. We conclude that $K_{a':b'} \not\to K_{a:b}$. $\square$

3 Mycielski graphs

Let G be a graph. The *Mycielski graph* $M(G)$ of G is obtained from G by for each vertex $v \in V(G)$, adding a new vertex c_v with the same neighbors, and then adding a vertex u with neighborhood $\{c_v : v \in V(G)\}$. If G is triangle-free, then $M(G)$ is also triangle-free. Furthermore, $\chi(M(G)) = \chi(G) + 1$.

Theorem 11. *Every graph G satisfies $\chi_f(M(G)) = \chi_f(G) + 1/\chi_f(G)$.*

Proof. Consider an $(a : b)$-coloring φ of G such that $\chi_f(G) = a/b$, which exists by Lemma 4. We will construct an $(a^2 + b^2 : ab)$-coloring ψ of $M(G)$, thus showing that $\chi_f(M(G)) \leq \frac{a^2+b^2}{ab} = a/b + b/a = \chi_f(G) + 1/\chi_f(G)$. Let $C = \{1\} \times [a]^2 \cup \{2\} \times [b]^2$; the coloring ψ will assign subsets of C of size ab to vertices of $M(G)$. For every $v \in V(G)$, let $\psi(v) = \{1\} \times \varphi(v) \times [a]$ and let $\psi(c_v) = \{1\} \times \varphi(v) \times [a - b] \cup \{2\} \times [b] \times [b]$. Let $\psi(u) = \{1\} \times [a] \times \{a - b + 1, \ldots, a\}$.

Let $w : V(G) \to \mathbf{R}_0^+$ be an assignment of weights to vertices of G such that $w(V(G)) = 1$ and $\alpha_w(G) = 1/\chi_f(G)$; such an assigment exists by Lemma 2 (scaling the assignment obtained by the lemma so that $w(V(G)) = 1$ if necessary). We will construct an assignment of weights $z : V(M(G)) \to \mathbf{R}_0^+$ such that $z(V(M(G))) = \chi_f(G)+1/\chi_f(G)$ and $\alpha_z(M(G)) \leq 1$, thus showing that $\chi_f(M(G)) \geq \frac{z(V(M(G)))}{\alpha_z(M(G))} \geq \chi_f(G) + 1/\chi_f(G)$. For each $v \in V(G)$, let $z(v) = (\chi_f(G) - 1)w(v)$ and $z(c_v) = w(v)$. Let $z(u) = 1/\chi_f(G)$. Consider now a maximal independent set I of $M(G)$:

- If $u \in I$, then $I \setminus \{u\}$ is a maximal independent set in G and $z(I) \leq (\chi_f(G) - 1)w(I) + 1/\chi_f(G) \leq 1$, since $w(I) \leq \alpha_w(I) = 1/\chi_f(G)$.
- Suppose now that $u \notin I$. Then $I = I_1 \cup I_2 \cup I_3$, where I_1 is an independent set in G, $I_2 = \{c_v : v \in I_1\}$, and $I_3 = \{c_v : v \in S\}$, where S is the set of vertices of G outside of the closed neighborhood of I_1. We have $\chi_f(G[S]) \leq \chi_f(G)$, and thus by Lemma 4, there exists an independent set $I_4 \subseteq S$ such that $w(I_4) \geq w(S)/\chi_f(G)$. Note that $I_1 \cup I_4$ is an independent set in G. Hence, we have $z(I) = (\chi_f(G) - 1)w(I_1) + w(I_1) + w(S) \leq \chi_f(G)(w(I_1) + w(I_4)) = \chi_f(G)w(I_1 \cup I_4) \leq 1$.

□

Note that if a and b are coprime, then $a^2 + b^2$ and ab are coprime. Hence, if $\chi_f(G) = a/b$, then $\chi_f(M(G)) = \frac{a^2+b^2}{ab}$ is a reduced fraction and its denominator is $ab > b^2$. Hence, by considering iterated Mycielski graphs of C_7, we have the following.

Corollary 12. *There exists a sequence of (triangle-free) graphs G_0, G_1, ... such that for $i \geq 0$, the graph G_i has $2^{i+3} - 1$ vertices and the denominator of $\chi_f(G_i)$ is at least $3^{2^i} > 3^{|G_i|/8}$.*

In particular, there are graphs whose optimal set coloring requires exponentially many colors.