Discharging and reducible configurations

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Suppose we want to show that graphs from some hereditary class $\mathcal{G}$ are k-colorable. Clearly, we can restrict our attention to graphs from $\mathcal{G}$ of minimum degree at least k: if a graph $G \in \mathcal{G}$ contains a vertex v of degree less than k, it suffices to find a k-coloring of the graph $G - v$ which also belongs to $\mathcal{G}$, and then give v a color different from the colors of its neighbors, which is always available since v has less than k neighbors.

More formally, we are constructing a proof by contradiction. Suppose that G is a graph belonging to $\mathcal{G}$ with the smallest number of vertices that is not k-coloriable. Hence, G is a hypothetical smallest counterexample to the claim that all graphs from $\mathcal{G}$ are k-colorable. By the previous paragraph, we see that G does not contain a vertex of degree at most $k - 1$. Substructures with the property that they cannot appear in any smallest counterexample are called *reducible configurations*, and they form a basis of many arguments in graph coloring.

In the most direct application, we identify a set of reducible configurations such that at least one of them must necessarily appear in any graph from $\mathcal{G}$, thus excluding the existence of a (smallest) counterexample and showing that indeed, all graphs from $\mathcal{G}$ are k-colorable. There are many tricks used in showing that a configuration is reducible. However, the basic idea almost always resembles our starting example: the configuration is removed (and possibly some further local changes are performed in the graph), and then a coloring of the modified graph is transformed into a coloring of the original graph. Reducible configurations usually involve vertices whose degree is smaller than or close to the number of colors k we are using, since a coloring of the rest of the graph can be extended to such vertices (nearly) greedily.

Consequently, reducible configurations are easiest to find in graph classes $\mathcal{G}$ that are sparse, in the sense that the graphs in $\mathcal{G}$ have average degree close to k. To prove that each graph in such a class $\mathcal{G}$ contains one of the reducible configurations, one needs to argue that their absence would force the graph to be too dense. The *discharging method* is a systematic way to organize such an argument. In the most straightforward form, a real number called *charge* is assigned to vertices (and in the case of embedded graphs, possibly faces) of the graph, so that the sum of the charges is negative; for example, if graphs in $\mathcal{G}$ have average degree less than d, a possible assignment of charge to a vertex v is $\deg(v) - d$. The charge is then redistributed according to some local rules, without changing its total amount. As a result, there exists a vertex (or face) whose final charge is negative, and based on the discharging rules, we argue that this is only possible if a reducible configuration appears in its neighborhood.

Initial assignment of charge

Lemma 1. *Let $a \geq 0$ and $b > 0$ be real numbers, and let G be a graph with 2-cell embedding in a surface of genus g. Let $ch_0(v) = a\deg(v) - 2b$ for $v \in V(G)$ and $ch_0(f) = (b-a)|f| - 2b$ for $f \in F(G)$. Then*

$$\sum_{v \in V(G)} ch_0(v) + \sum_{f \in F(G)} ch_0(f) = 2b(g-2).$$

Proof. By Euler's formula, we have

$$\begin{aligned}
\sum_{v \in V(G)} \mathrm{ch}_0(v) + \sum_{f \in F(G)} \mathrm{ch}_0(f) &= \left(-2b|V(G)| + a\sum_{v \in V(G)} \deg(v)\right) \\
&\quad + \left(-2b|F(G)| + (b-a)\sum_{f \in F(G)} |f|\right) \\
&= (-2b|V(G)| + 2a|E(G)|) + (-2b|F(G)| + 2(b-a)|E(G)|) \\
&= 2b(|E(G)| - |V(G)| - |F(G)|) = 2b(g-2).
\end{aligned}$$

□

The most common choices of charge for vertices v and faces f are the following:

- $\mathrm{ch}_0(v) = \deg(v) - 6$ and $\mathrm{ch}_0(f) = 2|f| - 6$, obtained by setting $a = 1$ and $b = 3$: For connected plane graphs, it sums to -12. The charge of all faces is nonnegative, putting the focus on vertices. This tends to be a good choice when proving results for unrestricted plane graphs.
- $\mathrm{ch}_0(v) = \deg(v) - 4$ and $\mathrm{ch}_0(f) = |f| - 4$, obtained by setting $a = 1$ and $b = 2$: For connected plane graphs, the charge sums to -8. Only 3-faces have negative charge, which makes this a good choice for triangle-free graphs or graphs with restrictions on triangles.

- $\mathrm{ch}_0(v) = 2\deg(v) - 6$ and $\mathrm{ch}_0(f) = |f| - 6$, obtained by setting $a = 2$ and $b = 3$: For connected plane graphs, the charge sums to -12. and this choice is convenient when dealing with 3-regular graphs (so that vertices have no charge), or sometimes when dealing with graphs of large girth.

Of course, other choices may be useful in other special situations. The proper choice of charge can greatly simplify the discharging rules. However, let us remark that at least in principle, all the charge choices are equivalent. Indeed, suppose that we have assigned charge $\mathrm{ch}_0(v) = a\deg(v) - 2b$ for $v \in V(G)$ and $\mathrm{ch}_0(f) = (b-a)|f| - 2b$ for $f \in F(G)$. Consider any other real numbers $a' \geq 0$ and $b' > 0$. Now, let each vertex send $\gamma = \frac{ab'-a'b}{b'}$ units of charge to each incident face, and let ch denote the resulting assigment of charges. We have $\mathrm{ch}(v) = (a-\gamma)\deg(v) - 2b = \frac{b}{b'}(a'\deg(v) - 2b')$ and $\mathrm{ch}(f) = (b-a+\gamma)|f| - 2b = \frac{b}{b'}((b'-a')|f| - 2b')$, which up to scaling is the same charge we would obtain if we instead of a and b chose a' and b'. Hence, if a discharging argument is possible with the latter choice, it is also possible with the former one, by adding the discharging rule of moving γ units of charge from each vertex to each incident face. In other words, choosing a "wrong" charge may make the discharging rules more complicated, but not impossible to find.

1 Easy example

Lemma 2. *Every planar graph without* 4*-cycles has a vertex of degree at most* 4*.*

Proof. Let G be a plane graph without 4-cycles. Without loss of generality, we can assume that G is connected and has at least three vertices; and consequently, no face of G has length 4.

Let us assign initial charge $\mathrm{ch}_0(v) = 2\deg(v) - 10$ to each vertex v of G, and initial charge $\mathrm{ch}_0(f) = 3|f| - 10$ to each face f of G. The sum of the initial charges is -20 by Lemma 1. Since G does not contain 4-cycles, no two triangles share an edge. Now, each (≥ 5)-face sends $1/3$ to each triangle with that it shares an edge. Let ch denote the final charge after performing the redistribution.

For each 3-face f, we have $\mathrm{ch}(f) = \mathrm{ch}_0(f) + 3 \times 1/3 = 0$. For a (≥ 5)-face f, we have $\mathrm{ch}(f) \geq \mathrm{ch}_0(f) - |f| \times 1/3 = 3|f| - 10 - |f|/3 = (8|f| - 30)/3 > 0$. Since the sum of the charges did not change by the redistribution, there exists a vertex with $\mathrm{ch}(v) = \mathrm{ch}_0(v) < 0$, and thus $\deg(v) < 5$. $\square$

Corollary 3. *Every planar graph without* 4*-cycles is* 4*-colorable.*

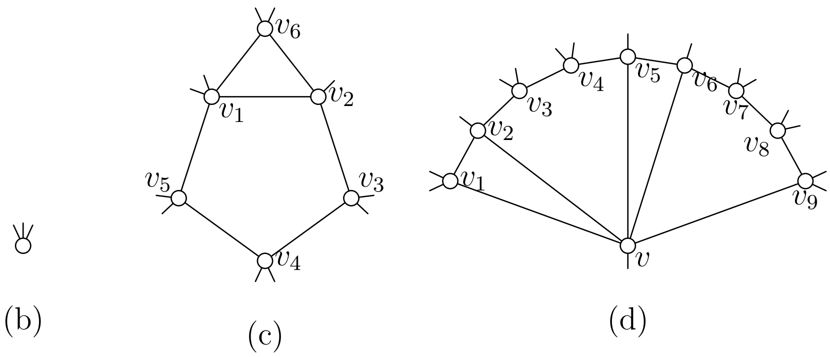

Figure 1: Reducible configurations from Lemma 4.

Proof. Let G be a plane graph without 4-cycles, and suppose for a contradiction that G is not 4-colorable. Let us choose such a graph G with the smallest number of vertices.

Let v be a vertex of G of minimum degree. By Lemma 2, we have $\deg(v) \leq 4$. Let φ be a 4-coloring of $G - v$. If $\deg(v) \leq 3$, then φ extends to a 4-colring of G greedily, which is a contradiction. Hence, suppose that $\deg(v) = 4$, and let $v_1, \ldots, v_4$ be neighbors of G in cyclic order accoring to the drawing of G. Since φ does not extend to a 4-coloring of G, we can assume that $\varphi(v_i) = i$ for $i \in \{1, \ldots, 4\}$. By planarity, G cannot contain both a path in colors 1 and 3 between v_1 and v_3, and a path in colors 2 and 4 between v_2 and v_4. By symmetry, we can assume that G does not contain a path in colors 1 and 3 between v_1 and v_3. Let H be the component of the subgraph of G induced by vertices of colors 1 and 3 that contains v_3. Exchanging colors 1 and 3 on the vertices of H and giving v the color 3 results in a proper 4-coloring of G, which is a contradiction. □

2 Discharging method example

We now give a further example demonstrating more of the aspects of the method. Specifically, we will show that all planar graphs without 4-cycles are actually 4-choosable. Let us start with the reducible configurations (see Figure 1).

Lemma 4. *Let G be a plane graph with no 4-cycles, such that every planar graph with no 4-cycles and with fewer than $|V(G)|$ vertices is 4-choosable. If G is not 4-choosable, then*

(a) G is connected;

(b) *the minimum degree of G is at least 4;*

(c) *if f is a 5-face of G sharing an edge v_1v_2 with a 3-face g and all vertices incident with f and g except for v_1 have degree 4, then v_1 has degree at least 6; and,*

(d) *if $v \in V(G)$ has degree 6 and is incident with a 3-face g_1, a 5-face f_1, a 3-face g_2, and a 5-face f_2 in order, then at least one vertex incident with these faces other than v has degree at least 5.*

Proof. Let L be any assignment of lists of size 4 to vertices of G.

For (a), if G were a disjoint union of non-empty graphs G_1 and G_2, then both G_1 and G_2 would be L-coloable by the assumption, and thus G would also be L-colorable. Similarly, for (b), if G contained a vertex v of degree at most 3, then $G - v$ is L-colorable by the assumption and the coloring can be extended to G by giving v a color in $L(v)$ different from the colors of its neighbors.

For (c), let $K = v_1v_2 \dots v_5$ be the boundary cycle of f. Since G has no 4-cycles, K is induced. Let $T = v_1v_2v_6$ be the triangle bounding the face g. Since K is an induced cycle, we have $v_6 \notin V(K)$, and since G contains no 4-cycles, v_6 has no neighbor in K other than v_1 and v_2. Suppose for a contradiction that $\deg(v_1) \leq 5$. By the assumptions, there exists an L-coloring φ of $G - \{v_1, \dots, v_6\}$. For $i \in [6]$, let $L'(v_i)$ denote the subset of $L(v_i)$ consisting of the colors not appearing on the neighbors of v_i in the coloring φ; we have $|L'(v_i)| \geq 2$ for $i \in \{1, 3, 4, 5, 6\}$ and $|L'(v_2)| \geq 3$. To extend the coloring φ to a 4-coloring of G, it suffices to L'-color $K \cup T$.

First, let us choose a color $c_1 \in L'(v_2)$ such that $|L'(v_3) \setminus \{c\}| \geq 2$. If we can color v_1 and v_6 by colors different from c_1, we can then color v_2 by c_1 and greedily color v_5, v_4, and v_3 in order by colors from their lists different from the colors of their neighbors. Otherwise, we have $L'(v_1) = L'(v_6) = \{c_1, c_2\}$ for some color c_2. Let $c_3 \in L'(v_2)$ be a color different from c_1 and c_2. We color v_2 by c_3 and greedily color v_3, v_4, v_5, v_1, and v_6 in order. It follows that G is 4-colorable, which is a contradiction; hence, (c) holds.

Finally, let us consider (d). Let vv_1v_2, $vv_2v_3v_4v_5$, vv_5v_6, and $vv_6v_7v_8v_9$ be the facial walks of g_1, f_1, g_2, and f_2. Observe that since G has no parallel edges, no 4-cycles, and minimum degree at least 4, the vertices v_1, …, v_9 are pairwise distinct. By the assumptions, there exists an L-coloring φ of $G - \{v, v_1, \dots, v_9\}$. Note that v has only one neighbor y colored by φ, and v_1 has exactly two neighbors y_1 and y_2 that are colored by φ (since v_1 cannot be adjacent to v_3, …, v_9 by the absence of 4-cycles). Hence, we can choose a color $c \in L(v) \setminus \{\varphi(y)\}$ such that $|L(v_1) \setminus \{c, \varphi(y_1), \varphi(y_2)\}| \geq 2$. Color v by the color c, then color vertices v_9, v_8, …, v_1 greedily in order using colors different from the colors of their neighbors colored earlier (note that even if there are some edges among $v_2, \ldots, v_9$ not contained in the path $v_2 \ldots v_9$, this does not affect the validity of the greedy coloring procedure). $\square$

Now, we can proceed with the discharging phase.

Theorem 5. *Every planar graph without* 4*-cycles is* 4*-choosable.*

Proof. Suppose for a contradiction this is not the case, and let G be a planar graph without 4-cycles with the smallest number of vertices such that G is not L-colorable for some assigment L of lists of size 4. By Lemma 4(a), G is connected. Let us assign initial charge $\mathrm{ch}_0(v) = \deg(v) - 4$ to each vertex v of G, and initial charge $\mathrm{ch}_0(f) = |f| - 4$ to each face f of G. The sum of the initial charges is -8 by Lemma 1.

Next, we redistribute the charge according to the following rules. For a 3-face g, let $r(g)$ denote the number of incident vertices of degree at least 5.

(R1) A vertex v of degree at least 5 incident with a 3-face g sends 1 to g if $\deg(v) \geq 7$, or if $\deg(v) = 6$ and v is incident with at most two triangles, or if $\deg(v) = 6$ and g is the only triagle incident with v satisfying $r(g) = 1$. The vertex v sends $2/3$ to g if $\deg(v) = 6$ and v is incident with three 3-faces g, g_1, and g_2 such that $r(g) = r(g_1) = 1$. Otherwise, v sends $1/2$ to g.

(R2) If a face f shares an edge uv with a 3-face g such that $\deg(u) = \deg(v) = 4$, then f sends $1/3$ to g if $r(g) = 0$, and f sends $1/6$ to g if $r(g) = 1$.

(R3) If a face f shares an edge uv with a 3-face g such that $r(g) = 1$ and $\deg(u) = 4$ and $\deg(v) \geq 5$, then f sends $1/6$ to g if $|f| \geq 6$, or $|f| = 5$ and $\deg(v) = 5$, or $|f| = 5$ and $\deg(v) = 6$ and f is incident with a vertex of degree at least 5 distinct from v.

Let ch denote the final charge after performing the redistribution. Note that the total amount of charge is unchanged, and thus the sum of final charges is negative.

We argue that each vertex and face has a non-negative final charge, which gives a contradiction. Let us start with vertices: By Lemma 4(b), each vertex v of G has degree at least 4. If $\deg(v) = 4$, then v sends no charge and $\mathrm{ch}(v) = \mathrm{ch}_0(v) = 0$. Suppose now that $\deg(v) \geq 5$, and let t denote the number of 3-faces incident with v. Since G contains no 4-cycles, no two 3-faces are consecutive in the cyclic order around v, and thus $t \leq \lfloor \deg(v)/2 \rfloor$. If $\deg(v) = 5$, then $t \leq 2$ and v sends $1/2$ to each incident 3-face by (R1), and thus $\mathrm{ch}(v) = \mathrm{ch}_0(v) - t/2 = 1 - t/2 \geq 0$. If $\deg(v) \geq 7$, then v sends 1 to each incident 3-face by (R1), and thus $\mathrm{ch}(v) = \mathrm{ch}_0(v) - t \geq \deg(v) - 4 - \lfloor \deg(v)/2 \rfloor = \lceil \deg(v)/2 \rceil - 4 \geq 0$. So, suppose that $\deg(v) = 6$. If $t \leq 2$, then $\mathrm{ch}(v) = \mathrm{ch}_0(v) - t \geq 0$. If $t = 3$, then let g_1, g_2, and g_3 be the 3-faces incident with v such that $r(g_1) \leq r(g_2) \leq r(g_3)$. If $r(g_2) \geq 2$, then v sends $1/2$ to g_2 and to g_3, and at most 1 to g_1, hence $\mathrm{ch}(v) \geq \mathrm{ch}_0(v) - 2 = 0$. If $r(g_2) = 1$, then v sends at most $2/3$ to each incident 3-face, and $\mathrm{ch}(v) \geq \mathrm{ch}_0(v) - 2 = 0$.

Next, consider the charge of the faces. Let g be a 3-face; since G does not contain 4-cycles and has minimum degree greater than 2, all faces that share an edge with g have length at least 5, and thus g sends no charge. If $r(g) \geq 2$, then g receives at least $1/2$ from each incident vertex of degree at least 5 by (R1) and thus g receives at least 1 in total. If $r(g) = 0$, then g receives $1/3$ from each face with that it shares an edge by (R2). Hence, suppose that $r(g) = 1$, and let v be the vertex of degree at least 5 incident with g. Note that g receives $1/6$ from the face opposite to v by (R2). If $\deg(v) = 5$, then g receives $1/2$ from v by (R1) and $1/6$ from each of the two faces with that g shares an edge incident with v by (R3), and the total amount received is $1/2 + 3 \times 1/6 = 1$. If $\deg(v) = 7$, or $\deg(v) = 6$ and v is inicident with at most two 3-faces, or $\deg(v) = 6$ and g is the only 3-face with $r(g) = 1$ incident with v, then g receives 1 from v by (R1). Finally, suppose that $\deg(v) = 6$, v is incident with three 3-faces, and a 3-face $g_1 \neq g$ incident with v satisfies $r(g_1) = 1$. In this case, v sends $2/3$ to g. Let f_1, g, f_2, g_1 be faces incident with v in order according to the drawing of G. By Lemma 4(d), it cannot be the case that both f_1 and f_2 are 5-faces whose only incident vertex of degree at least 5 is v, and thus either f_1 or f_2 sends $1/6$ to g by (R3). The total amount sent to g is at least $2/3 + 2 \times 1/6 = 1$. In all the cases, the final charge of g is at least $\mathrm{ch}_0(g) + 1 = 0$.

Finally, let f be a face of length at least 5. If $|f| \geq 6$, then $\mathrm{ch}(f) \geq \mathrm{ch}_0(f) - |f|/3 = \frac{2}{3}|f| - 4 \geq 0$ by (R2) and (R3). Hence, suppose that $|f| = 5$. If f sends at most $1/6$ to each incident 3-face by (R2), then $\mathrm{ch}(f) \geq \mathrm{ch}_0(f) - 5/6 > 0$. Hence, suppose that f sends $1/3$ to an incident 3-face g with $r(g) = 0$ by (R2). By Lemma 4(c), not all vertices incident with f have degree 4. Let $v_1v_2v_3v_4v_5$ be the boundary walk of f, where $\deg(v_5) \geq 5$. If at least one of $v_1, \ldots, v_4$ has degree at least 5, then f sends at most $\max(2 \times 1/3 + 2 \times 1/6, 1/3 + 4 \times 1/6) = 1$ by (R2) and (R3), and $\mathrm{ch}(f) \geq \mathrm{ch}_0(f) - 1 = 0$. Hence, suppose that $\deg(v_1) = \ldots = \deg(v_4) = 4$. Let g_1 and g_2 be the faces sharing edges v_1v_5 and v_4v_5 with f, respectively. If f sends charge to neither g_1 nor g_2 by (R3), then f sends at most $3 \times 1/3$ by (R2) and (R3), and $\mathrm{ch}(f) \geq \mathrm{ch}_0(f) - 1 = 0$. By symmetry, we can assume that f sends positive amount of charge to g_1, and thus g_1 is a 3-face with $r(g_1) = 1$. Since the only vertex incident with f of degree at least 5 is v_5, the rule (R3) applies only if $\deg(v_5) = 5$. However, this is not possible by Lemma 4(c). $\square$